FINDING INVARIANTS OF GROUP ACTIONS ON FUNCTION SPACES, A GENERAL METHODOLOGY FROM NON-ABELIAN HARMONIC ANALYSIS

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Neither dedicated to S. Royal, nor dedicated to N. Sarkozy

Abstract. In this paper, we describe a general method using the abstract non-Abelian Fourier transform to construct "rich" invariants of group actions on functional spaces.

In fact, this method is inspired of a classical method from image analysis: the method of Fourier-Descriptors, for discrimination among "contours" of objects. This is the case of the Abelian circle group, but the method can be extended to general non-Abelian cases.

Here, we improve on some of our previous developments on this subject, in particular in the case of compact groups and motion groups. The last point (motion groups) is in the perspective of invariant image analysis. But our method can be applied to many practical problems of discrimination, or detection, or recognition.

1. Introduction

In the paper, we consider the very general problem of finding "rich" invariants of the action of a (locally compact) group G on functions over G or over one of its homogeneous spaces. We start from a very old idea coming from a classical engineer's technique for invariant objects recognition: the Fourier-descriptors method. Invariant objects recognition is a critical problem in image processing. To solve it, numerous approaches have been proposed in the literature, often based on the computation of invariants followed by a classification method. Considering the group of motions of the plane, Gauthier and al. [9], [12], proposed a family of invariants, called Motion Descriptors, which are invariants in translation, rotations, scale and reflections. H. Fonga [7] applied them to grey level images. A recent survey on this question can be found in [20]. Another interesting paper closely connected to this work is [16].

In this paper, we develop and we give final results of a general theory of "Fourier-Descriptors". The paper contains really new results that justify the choice of these "Generalized Fourier Descriptors".

The paper deals mostly with two cases: first, the case of compact groups, and second, the case of certain "Motion groups", for the purpose of image analysis.

In another forthcoming paper [18], we show application of our results to several problems of pattern recognition (in particular, human-face recognition). In this

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last paper, a main point is that we apply 2D-invariant Motion-Descriptors for 3D recognition. The justification is clear: in practice we get a number of 2D images of the same object under several points of view. The Motion Descriptors being motion-invariants, we need a single picture for each point of view, independently of the position of the object. Also, in this paper, we use the invariants in the context of a classifier of Support-Vector-Machine type [21].

However, in another practical context (3D data for instance) we could apply our methodology to the action of the group $SO_3 \ltimes \mathbb{R}^3$ of 3D-motions. Generalized Motion Descriptors for this group action can be computed easily using our theory.

We obtained a long time ago the results presented here in the case of compact groups. But proofs of them were never published. We give these proofs here (Theorem 5). Our final (original) result (in the case of the discrete 2-D motion groups acting on the plane) is stated and proved in Theorem 8.

Along the paper, we use the terminologies "Fourier Descriptors", or "Generalized Fourier Descriptors" for general groups. When we want to focus on pattern recognition and motion groups, we use the terminology "Motion Descriptors".

1.1. **Review of known Motion Descriptors for plane images.**

1.1.1. *Definition of First-Type-Motion-Descriptors.* First-Type-Motion-Descriptors (1^{st}MD) are defined as follows. Let f be a square summable function on the plane, and $\tilde{f}$ its Fourier transform[1]:

$$\tilde{f}\,(\xi) = \int_{\mathbb{R}^2} f\,(x)\, e^{-i\langle x,\xi\rangle_{\mathbb{R}^2}}\, dx \tag{1.1}$$

If (λ, θ) are polar coordinates of the point ξ, we shall denote again by $\tilde{f}(\lambda, \theta)$ the Fourier transform of f at the point (λ, θ). We define ([12], [9]) the mapping:

$$\begin{array}{rcl} I_1^r(f) & : & \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \\ r & \longrightarrow & I_1^r(f), \end{array}$$

by

$$I_1^r(f) = \int_0^{2\pi} \left|\tilde{f}\,(\lambda, \theta)\right|^2 d\theta \tag{1.2}$$

Here I_1^r is the feature vector which describes each image f and will be used as an input of our first supervised classification method.

1.1.2. *Properties.* Fourier descriptors I_1^r calculated according to equation (1.2), have several elementary properties crucial for invariant object recognition [9]:

Motion-Descriptors are motion and reflection-invariant:

- If M is a “Motion” such as $g = f \circ M$,

$$I_1^r(f) = I_1^r(g), \forall r \in \mathbb{R}^+ \tag{1.3}$$

- If there exists a reflections $\Re$ such that $g = f \circ \Re$,

[1]All along the paper, we omit the important detail that certain formulas make sense in fact on $\mathbb{L}^1 \cap \mathbb{L}^2$ spaces only, but prolong in a unique way to $\mathbb{L}^2$ spaces. It is the case here.

$$I_1^r(f) = I_1^r(g), \forall r \in \mathbb{R}^+ \tag{1.4}$$

- Motion descriptors are scaling-covariant:

If k is a real constant such as $g(x) = f(kx)$ for all $x \in \mathbb{R}^2$,

$$I_1^r(g) = \frac{1}{k^4} I_1^{\frac{r}{k}}(f), \forall r \in \mathbb{R}^+. \tag{1.5}$$

The proof is obvious and left to the reader.

1.1.3. *Definition of Second-Type "Motion Descriptors"* . Second-Type "Motion-Descriptors" (2^{nd} MD) are a second family of invariants (containing the first one) which is "closer to completeness" and completely natural as explained in the second part of this paper. Originally they were defined in [11] and [7]. They are denoted by I^{ξ_1,ξ_2} and they are defined by:

$$\begin{aligned} I^{\xi_1,\xi_2}(f) &= \int_{S_1} \tilde{f}(R_\theta(\xi_1+\xi_2) \times \\ &\quad \overline{\tilde{f}}(R_\theta(\xi_1))\overline{\tilde{f}}(R_\theta(\xi_2))d\theta, \\ \xi_1,\xi_2 &\in \mathbb{R}^2. \end{aligned} \tag{1.6}$$

Here $R_\theta(\xi)$ denotes the rotation of angle θ of the vector $\xi \in \mathbb{R}^2$, i.e. $R_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

Remark 1. *1. It is clear that I^{ξ_1,ξ_2} is invariant with respect to motions.*
2. It is also clear that the set of invariants I^{ξ_1,ξ_2} is completely determined by the smaller set obtained by taking ξ_1 of the form $(0,r_1)$, $r_1 \in \mathbb{R}^+$.

Hence an alternative definition of I^{ξ_1,ξ_2} is given by:

$$\begin{aligned} I_f^w(\lambda_1,\lambda_2) &= \int_{S_1} [\tilde{f}(-\lambda_1\sin(\theta+\omega) - \lambda_2\sin\theta, \\ &\quad \lambda_1\cos(\theta+\omega) + \lambda_2\cos\theta) \\ &\quad \overline{\tilde{f}}(-\lambda_1\sin(\theta+\omega), \lambda_1\cos(\theta+\omega)) \\ &\quad \overline{\tilde{f}}(-\lambda_2\sin\theta, \lambda_2\cos(\theta))]d\theta, \end{aligned} \tag{1.7}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}^+$ and $\omega \in [0, 2\pi[$.

1.1.4. *Properties.* The following properties are elementary and left to the reader to check:

- For a real-valued f, $I^{\xi_1,\xi_2}(f)$ is a real number.
- The quantity $I^{\xi_1,\xi_2}(f)$ is symmetric in ξ_1, ξ_2, i.e.:

$$I_f^w(\lambda_1,\lambda_2) = I_f^{-w}(\lambda_1,\lambda_2) = I_f^w(\lambda_2,\lambda_1) \tag{1.8}$$

2. Theory of Generalized Fourier Descriptors

2.1. **Organization.** -Section 2.2:

We first recall the beginning of the story of Fourier-Descriptors, namely descriptors for "exterior-contours" of objects. They give rise to a set of complete invariants of exterior-contours that are denoted here by P_n and $R_{n,m}$. The invariant P_n is a spectral density, while $R_{n,m}$ is homogeneous to a phase.

We recall the definition of general Fourier transforms on topological groups with their main property.

For the purpose of generalization, we change a bit the class of phase-invariants, and we replace it by the other phase invariants $\tilde{R}_{n,m}$, more or less equivalent. At this step, we are able to find a complete and natural generalization of the invariants, in terms of Fourier-transforms of functions (images) on any unimodular locally-compact group G.

We call these two families of invariants the first and second-Type "Fourier Descriptors", (or Motion-descriptors when G is a group of motions). Again, first-type invariants are homogeneous to spectral-densities, and second-type are "phase-invariants".

-Section 2.3:

We make the explicit computation of the Motion-descriptors (first and second-type) in the case of the group M_2 of motions (rotations+translations) of the Euclidian plane $\mathbb{R}^2$. To do this, we consider images (i.e. functions on the plane) as functions on the group by considering these functions as independent of the rotation angle. We call this "forgetting" operation the "trivial lift" (of an image on the plane to an image on M_2).

-Section 3:

We show that, in the case of **any compact group** G, our first and second-Type Fourier-Descriptors are weakly-complete (i.e. They separate the functions on a very big ("residual") subset of the set of images over G). The case of "exterior-contours" of objects is just the special case of the compact "circle" group C.

To do this, we need the Tannaka-Krein theory, and we present it under the form of Chu-Theory, which is a generalization that we need later.

-Section 4 is devoted to the case of the Motion-Descriptors in the case of the groups $M_{2,N}$ (i.e. the groups of motions with discrete rotations of elementary angle $\frac{2\pi}{N}$), for N an **odd** integer. This is the hardest part of our work. We compute Motion-Descriptors in this case, and we conclude to their weak-completeness, using deeply the theory of Moore-groups, and the Chu-Duality.

2.2. **Preliminaries.** The Fourier-Descriptors method is a very old method used for pattern analysis. The oldest reference we were able to find is [17]. A recent one is [20]. One of the authors and his co-workers have several contributions in the area ([9], [11], [7], [12]). Basically, the method uses the good properties of standard Fourier series with respect to translations. For the sake of completeness, let us recall this basic old naive idea, that has been used successfully several times for pattern recognition. For details, see for instance [17].

The method applies to the problem of discrimination of 2D-patterns by their **exterior** contour. Let the exterior contour be well defined, and regular enough (piecewise smooth, say). Assume that it is represented as a closed curve, arclength parametrized from some initial point θ_0 on the curve and denoted by $s(\theta)$. By construction, the function $s(\theta)$ is obviously invariant under 2D translation of the pattern. Let now $\hat{s}_n$ denote the Fourier series of the periodic function $s(\theta)$. The only arbitrary object that makes the function s non-invariant under motions (translations plus rotations) of the pattern, is the choice of the initial point θ_0. As it is well known, a translation of θ_0 by a, $\theta_0 := a + \theta_0$, changes $\hat{s}_n$ for $e^{ian}\hat{s}_n$, where $i = \sqrt{-1}$. (Here, the total arclength is normalized to 2π). Set $\hat{s}_n = \rho_n e^{i\varphi_n}$. Let us define the "ratios of phases" $R_{n,m} = \frac{\varphi_n}{n} - \frac{\varphi_m}{m}$ Then, it is easy to check that the "discrete power spectral densities" $Pn = |\hat{s}_n|^2$ and the "ratios of phases" $R_{n,m}$ **form a complete set of invariants** of exterior contours, under motions of the plane. They are also homothetic-invariants as soon as the arclength is normalized.

This result is extremely efficient for shape discrimination, it has been used an incredible number of times in many areas, and it is very robust and physically interesting for several reasons (in particular the fact that the P_n are just discrete "power spectral densities", and that both P_n and $R_{n,m}$ can be computed very quickly using FFT algorithms). Also, the extraction of the "exterior-contour" is more or less a standard procedure in image processing.

The main default of the method is that it doesn't take any account of the "texture" of the pattern: two objects with similar exterior contours have similar "Fourier-Descriptors" P_n and $R_{n,m}$.

As soon as one knows a bit about abstract harmonic analysis, one immediately thinks about possible abstract generalizations of this method. The first paper that we know in which this idea of "abstract generalization" of the method appears is the paper [4]. One of the authors worked on the subject, with several co-workers ([9], [11], [7], [12]). In particular, there is a lot of very interesting results in the theses [11] and [7]. Unfortunately, these results being very incomplete, they were never completely published. We would like here to give a series of more or less final result, not yet completely satisfactory, but very interesting and convincing.

2.2.1. *First preliminary: The Fourier Transform on locally compact unimodular groups.* Classical Fourier descriptors for exterior contours will just correspond to the case of the "circle" group, as the reader can check, i.e. the group of rotations $e^{i\theta}$ of the complex plane.

By a famous theorem of Weil, a locally compact group possesses a (almost unique) Haar-measure ([24]), i.e. a measure which is invariant under (left or right) translations. For instance the Haar measure of the circle group is $d\theta$ since $d(\theta+a) = d\theta$. A group is said unimodular if its left and right Haar measures can be taken equal (that is, the Haar measure associated with left or right translations). An abelian group is obviously automatically unimodular. A less obvious result is that a compact group is automatically unimodular.

The most pertinent examples for pattern recognition are of course the following:
1. The circle group C.

2. The group of motions of the plane M_2. It is the group of rotations and translations (θ, x, y) of the plane. As one can check, the product law on M_2 is

$$(\theta_1, x_1, y_1).(\theta_2, x_2, y_2) = \tag{2.1}$$
$$(\theta_1 + \theta_2, \cos(\theta_1)x_2 - \sin(\theta_1)y_2 + x_1, \sin(\theta_1)x_2 + \cos(\theta_1)y_2 + y_1).$$

It represents the geometric composition of two motions. The main difference with the circle group is that it is not Abelian (commutative). This expresses the fact that rotations and translations of the plane do not commute. However, it is unimodular: the measure $d\theta dx dy$ is left and right invariant, hence it is the Haar measure.

3. The group of $y-$homotheties and $x-$translations of the upper two dimensional half plane: $(y_1, x_1).(y_2, x_2) = (y_1 y_2, x_1{+}x_2)$. Here, the $y_i' s$ are positive real numbers. Left and right Haar measure is $dx\frac{dy}{y}$ since $dx\frac{dy}{y} = d(x+a)\frac{d(by)}{by}$.

This abelian group is related to the classical Fourier-Mellin transform. A similar group of interest is the (abelian) group of θ-rotations and λ homotheties of the complex or two dimensional plane: $(\theta_1, \lambda_1).(\theta_2, \lambda_2) = (\theta_1 + \theta_2, \lambda_1 \lambda_2)$. Here again, the λ_i's are positive real numbers but the θ_i's belong to the circle group. Of course, if one takes an image centered around it's gravity center, then, the effect of translations is eliminated, and it remains only the action of rotations and homotheties. Applying the theory developed in the second part of this paper to the case of this group leads to complete invariants with respect to motions and homotheties. This is related with the nice work of [10].

Unfortunately, in this case, the computation of all the invariants is based upon a preliminary estimation of the gravity center of the image. Hence, the invariants are simultaneously very sensitive to this preliminary estimation.

4. The group of translations, rotations and homotheties of the 2D plane itself (we don't write the multiplication but it is obvious) is unfortunately not unimodular. Hence the theory in this Section does not apply. It is why one has to go back to the previous group.

5. The group SO_3 of rotations of $\mathbb{R}^3$. It is related to the human biological mechanisms of pattern recognition (see the paper [4]).

6. Certain rather **unusual groups** play a fundamental role in our theory below: the groups $M_{2,N}$ of motions, the rotation component of which is an integer multiple of $\frac{2\pi}{N}$. They are subgroups of M_2, and if N is large, $M_{2,N}$ could be reasonably called the "group of translations and sufficiently small rotations". In some precise mathematical sense, M_2 is the limit when N tends to infinity of the groups $M_{2,N}$.

For standard Fourier series and Fourier transforms, there are several general ingredients. Fourier series correspond to the circle group, Fourier transforms to the $\mathbb{R}$ (or more generally $\mathbb{R}^p$) group. In both cases, we have the formulas:

$$\begin{aligned} \hat{s}_n &= \int_G s(\theta)\, e^{-in\theta}\, d\theta \\ \hat{f}(\lambda) &= \int_G f(x)\, e^{-i\lambda x}\, dx. \end{aligned} \tag{2.2}$$

Formally, in these two formulas appear an integration over the group G with respect to the Haar measure (respectively $d\theta, dx$) of the function (respectively s, f)

times (the inverse of) the "mysterious" term $e^{in\theta}$ (resp. $e^{i\lambda x}$). This term is the "character" term. It has to be interpreted as follows: For each n (resp. λ), the map $\mathbb{C} \to \mathbb{C}$, $z \to e^{in\theta}z$ (resp. the map $z \to e^{i\lambda x}z$) is a unitary map (i.e. preserving the norm over $\mathbb{C}$), and the map $\theta \to e^{in\theta}$ (resp. $x \to e^{i\lambda x}$) is a continuous[2] group-homomorphism to the group of unitary linear transformations of $\mathbb{C}$. For a general topological group G, such a mapping is called a "character" of G.

The main basic result is the Pontryagin's duality theorem, that claims the following:

Theorem 1. *(Pontryagin's duality Theorem) The set of characters of an Abelian locally-compact group G is a locally-compact abelian group (under natural multiplication of characters), denoted by $G\hat{}$, and called the dual group of G. The dual group $(G\hat{})\hat{}$ of $G\hat{}$ is isomorphic to G.*

Then, the Fourier transform over G is defined like that: it is a mapping from $\mathbb{L}^2(\mathrm{G},dg)$ (space of square integrable functions over G, with respect to the Haar measure dg), to the space $\mathbb{L}^2(\mathrm{G}\hat{}, d\hat{g})$, where $d\hat{g}$ is the Haar measure over $G\hat{}$:

$$\begin{aligned} f &\to \hat{f}, \\ \hat{f}(\hat{g}) &= \int_G f(g)\chi_{\hat{g}}(g^{-1})dg. \end{aligned} \tag{2.3}$$

Here, $\hat{g} \in G\hat{}$ and $\chi_{\hat{g}}(g)$ is the value of the character $\chi_{\hat{g}}$ on the element $g \in G$.

As soon as one knows that the dual group of $\mathbb{R}$ is $\mathbb{R}$ itself, and the dual group of the circle group is the discrete additive group $\mathbb{Z}$ of integer numbers, it is clear that Formulas 2.2 are particular cases of Formula 2.3.

It happens that there is a generalization of the usual **Plancherel's Theorem**: The Fourier Transform[3] is an isometry from $\mathbb{L}^2(\mathrm{G}, dg)$ to $\mathbb{L}^2(\mathrm{G}\hat{}, d\hat{g})$. The general form of the inversion formula follows:

$$f(g) = \int_G f(\hat{g})\chi_{\hat{g}}(g)d\hat{g}. \tag{2.4}$$

In our cases ($\mathbb{R}$,C), this gives of course the usual formulas.

In the case of nonabelian groups, the generalization starts to be less straightforward. To define a reasonable Fourier transform, one cannot consider only characters (this is not enough for a good theory, leading to Plancherel's Theorem). One has to consider more general objects than characters, namely, unitary irreducible representations of G. A (continuous) unitary representation of G consists of replacing $\mathbb{C}$ by a general complex Hilbert[4] space H, and the characters $\chi_{\hat{g}}$ by unitary linear operators $\chi_{\hat{g}}(g) : H \to H$, such that the mapping $g \to \chi_{\hat{g}}(g)$ is a continuous[5]

[2]Along the paper, the topology over unitary operators on a Hilbert or Euclidian space is not the normic, but the stong topology.

[3]Precisely, Haar measures can be normalized so that Fourier transform is isometric.

[4]In the paper, all Hilbert spaces are assumed separable.

[5]For the strong topology of the unitary group $U(H)$.

homomorphism. Irreducible means that there is no nontrivial closed subspace of H which is invariant under all the operators $\chi_{\hat{g}}(g)$, $g \in G$. Clearly, characters are very special cases of continuous unitary irreducible representations. The main fact is that, for locally compact nonabelian groups, to get Plancherel's formula, it is enough to replace characters by these representations.

Definition 1. *Two representations χ_1, χ_2 of G, with respective underlaying Hilbert spaces H_1, H_2 are said equivalent if there is a linear invertible operator $A : H_1 \rightarrow H_2$, such that, for all $g \in G$:*

$$\chi_2(g) \circ A = A \circ \chi_1(g). \tag{2.5}$$

More generally, a linear operator A, eventually noninvertible, meeting condition 2.5, is called an intertwining operator between the representations χ_1, χ_2.

The set of equivalence classes of unitary irreducible representations of G is called the dual set of G, and is denoted by $G^{\hat{}}\cdot$

One of the main differences with the abelian case is that $G^{\hat{}}$ has in general no group structure. However, in this very general setting, Plancherel's Theorem holds:

Theorem 2. *Let G be a locally compact unimodular, type 1 group[6] with Haar measure dg. Let $G^{\hat{}}$ be the dual of G. There is a measure over $G^{\hat{}}$ (called the Plancherel's measure, and denoted by $d\hat{g}$), such that, if we define the Fourier transform over G as the mapping:*

$$\begin{aligned} \mathbb{L}^2(G,dg) &\rightarrow \mathbb{L}^2(G^{\hat{}},d\hat{g}), \\ f &\rightarrow \hat{f}, \\ \hat{f}(\hat{g}) &= \int_G f(g)\chi_{\hat{g}}(g^{-1})dg, \end{aligned} \tag{2.6}$$

then, $\hat{f}(\hat{g})$ is a Hilbert-Schmidt operator over the underlaying space $H_{\hat{g}}$, and the Fourier transform is an isometry.

As a consequence, the following inverse formula holds:

$$f(x) = \int_{G^{\hat{}}} Trace[\hat{f}(\hat{g})\chi_{\hat{g}}(g)]d\hat{g}. \tag{2.7}$$

More generally, if χ is a unitary representation of G **-not necessarily irreducible-** one can define the Fourier transform $\hat{f}(\chi)$ by the same formula 2.6.

All this could look rather complicated. In fact, it is not at all, and we shall immediately make it explicit in the case of main interest for our applications to pattern recognition, namely the group of motions M_2.

In the following, for the group M_2, (and later on $M_{2,N}$), we take up the notations below:

[6] A locally compact group has type1 if all its unitary representations admit an irreducible desintegration. All groups under consideration in this paper are type 1.

Notation 1. *Elements of the group are denoted indifferently by* $g = (\theta, x, y) = (\theta, X)$*, where* $X = (x, y) \in \mathbb{R}^2$*. The usual scalar product over* $\mathbb{R}^2$ *is denoted by* $< ., . >_{\mathbb{R}^2}$*, or simply* $< ., . >$ *if no confusion is possible. Then, the product over* M_2 *(resp.* $M_{2,N}$ *writes* $(\theta, X).(\alpha, Y) = (\theta + \alpha, R_\theta Y + X)$*, where* R_θ *is the rotation of angle* θ*.*

Example 1. *Group* M_2 *of motions of the plane.*

-In that case, the unitary irreducible representations fall in two classes: 1. characters (one dimensional Hilbert space of the representation), 2. The other irreducible representations have infinite dimensional underlaying Hilbert space $H = \mathbb{L}^2(C, d\theta)$ *where* C *is the circle group* $\mathbb{Z}/2\pi\mathbb{Z}$*, and* $d\theta$ *is the Lebesgue measure over* C*. These representations are parametrized by any ray* R *from the origin in* $\mathbb{R}^2$*,* $R = \{\alpha V$*,* V *some fixed nonzero vector in* $\mathbb{R}^2$*,* α *a real number,* $\alpha > 0\}$*. For* $r \in R$ *(the ray), the representation* χ_r *expresses as follows, for* $\varphi(.) \in H$*:*

$$[\chi_r(\theta, X).\varphi](u) = e^{i<r, R_u X>}\varphi(u + \theta). \tag{2.8}$$

The Plancherel's measure has support the second class of representations, and is just the Lebesgue measure over the ray R*.*

It is easily computed that the Fourier transform of $f \in \mathbb{L}^2(M_2, Haar)$ *writes, with* $X = (x, y)$*:*

$$\begin{aligned}[\hat{f}(r).\varphi](u) &= \int\int\int_{M_2} f(\theta, x, y) e^{-i<r, R_{u-\theta} X>} \\ &\quad \times \varphi(u - \theta) d\theta dx dy.\end{aligned} \tag{2.9}$$

Remark 2. *Working a bit with the inverse formula 2.7 shows easily that this Fourier-transform on* M_2 *is closely related with the usual Fourier-Bessel transform. Another way to see this is to take for* φ *an element of the orthonormal basis of* H*,* $\{e^{in\theta}, n \in \mathbb{Z}\}$*. With* $r = (0, a)$*, if we take* f *as a function of* (x, y) *only, and if* $x = \lambda\cos(\omega), y = \lambda\sin(\omega)$*, one obtains after very elementary computations that Formula 2.9 can be rewritten as:*

$$[\hat{f}(r).\varphi](u) = i^n e^{inu} \int\int_{\mathbb{R}^2} f(\lambda, \omega) J_n(\lambda a) \lambda d\lambda d\omega, \tag{2.10}$$

where J_n *is the* n^{th} *Bessel function. This is the usual formula for the Fourier-Bessel transform.*

The main property of the general Fourier-transform that we will use in the paper concerns obviously its behavior with respect to translations of the group. Let $f \in \mathbb{L}^2(\mathrm{G}, dg)$ and set $f_a(g) = f(ag)$. Due to the invariance of the Haar measure w.r.t. translations of G, we get the **crucial** generalization of a well known formula:

$$\hat{f}(\hat{g}) \circ \chi_{\hat{g}}(a) = \widehat{f_a}(\hat{g}). \tag{2.11}$$

2.2.2. *Second preliminary: general definition of the Generalized Fourier Descriptors, from those over the circle group* . In the case of exterior contours of 2D patterns, the group under consideration is the circle group C. The set of invariants $P_n, R_{m,n}$ has first to be replaced by the (**almost equivalent**) set of invariants, $P_n, \tilde{R}_{m,n}$, where the new "phase invariants" $\tilde{R}_{m,n}$ are defined by:

$$\tilde{R}_{m,n} = \hat{s}_n \hat{s}_m \overline{\hat{s}_{n+m}}. \tag{2.12}$$

The first 3 lemmas (9, 10, 11) of Appendix 5 justify this definition: at least on a residual subset of $\mathbb{L}^2(C)$, these sets of invariants are equivalent. This is enough for our practical purposes.

Remark 3. *1. There is a counterexample in* [12] *showing that the second set of invariants is weaker (does not discriminate among all functions).*

But in practice, discriminating over a very big dense subset of functions is enough.

2. For the purpose of a generalization to more complicated situations, (and general in the category of locally compact groups), it is not reasonable to expect that a small simple set of invariants will discriminate among all the orbits of the action of a very small group on a large space.

3. Nevertheless, in the case of the additive groups $\mathbb{R}^n$, these second invariants discriminate completely. This is shown in [11].

4. For complete invariants over $\mathbb{L}^2(G)$ in ***the general abelian case****, generalizing those, see* [12], [11], [7].

Now, an important fact has to be pointed out. There is a natural interpretation and generalization of the "phase-invariants" $\tilde{R}_{m,n}$ in terms of representations.

We are given an arbitrary unimodular group G, with Haar measure dg. We define the Fourier transform $\hat{f}$ of f, as the map from the set of (equivalence classes of) unitary irreducible representations of G, defined by formula 2.6.

Let us state now a crucial definition, and a crucial theorem.

Definition 2. *The following sets I_1, I_2, are called respectively the first and second-Fourier-Descriptors (or Motion-Descriptors) of a map $f \in \mathbb{L}^2(G)$. For $\hat{g}, \hat{g}_1, \hat{g}_2 \in G\hat{}$,*

$$\begin{aligned} I_1^{\hat{g}}(f) &= \hat{f}(\hat{g}) \circ \hat{f}(\hat{g})^*, \\ I_2^{\hat{g}_1,\hat{g}_2}(f) &= \hat{f}(\hat{g}_1)\hat{\otimes}\hat{f}(\hat{g}_2) \circ \hat{f}(\hat{g}_1\hat{\otimes}\hat{g}_2)^*, \end{aligned} \tag{2.13}$$

where $\hat{f}(\hat{g})^*$ denotes the adjoint of $\hat{f}(\hat{g})$,

and where $\hat{g}_1\hat{\otimes}\hat{g}_2$ denotes the (equivalence class of) (Kronecker) Hilbert tensor product of the representations $\hat{g}_1$ and $\hat{g}_2$, and $\hat{f}(\hat{g}_1)\hat{\otimes}\hat{f}(\hat{g}_2)$ is the Hilbert tensor product of the Hilbert-Schmidt operators $\hat{f}(\hat{g}_1)$ and $\hat{f}(\hat{g}_2)$.

Then, clearly, in the particular case of the circle group, **these formulas coincide** with those defining P_n, $\tilde{R}_{m,n}$.

Theorem 3. *A (grey-level) image on G is a compactly supported real nonzero function over G, with positive values (the grey levels).*

Then, for images f, $I_1(f)$ is determined by $I_2(f)$ (by abuse, we write $I_1(f) \subset I_2(f)$) and $I_1(f), I_2(f)$ are invariant under translations of f by elements of G.

Proof. That $I_1(f)$ is determined by $I_2(f)$ comes from the fact that, f being an image, taking for $\hat{g}_2$ the trivial character c_0 of G, we get that $I_2^{\hat{g}_1,\hat{g}_2}(f) = av(f)I_1^{\hat{g}_1}(f)$, where the "mean value" of $f, av(f) = \int_G f(g)dg > 0$, $av(f) = (I_2^{c_0,c_0})^{1/3}$. That $I_1^{\hat{g}}(f_a) = I_1^{\hat{g}}(f)$ (where $f_a(g) = f(ag)$, the translate of f by a) comes from the classical property 2.11 of Fourier transforms. That $I_2^{\hat{g}_1,\hat{g}_2}(f) = I_2^{\hat{g}_1,\hat{g}_2}(f_a)$, comes from the other trivial fact, just a consequence of the definition,

$$\hat{f}_a(\hat{g}_1\hat{\otimes}\hat{g}_2) = \hat{f}(\hat{g}_1\hat{\otimes}\hat{g}_2) \circ (\chi_{\hat{g}_1}(a)\hat{\otimes}\chi_{\hat{g}_2}(a)),$$

and from the unitarity of the representations. □

Our purpose in the remaining of the paper is to compute and to investigate about the completeness (at least on a big subset of $\mathbb{L}^2(G)$) and the pertinence of these invariants, in the case of a general G, and specially in the case of our motion groups M_2 and $M_{2,N}$.

2.3. **The generalized Fourier Descriptors for the motion group M_2.** Here, using the results stated in Example 1, let us compute the generalized Fourier Descriptors, and observe that **these invariants coincide with the invariants under consideration in the first part of this paper.**

The following series of formulas comes from straightforward computations, using the results and notations stated in Example 1.

For $r_1, r_2 \in R$ (the ray), the tensor product $\chi_{r_1}\hat{\otimes}\chi_{r_2}$, (denoted also by $\chi_{r_1\hat{\otimes}r_2}$) of the representations χ_{r_1} and χ_{r_2} can be written, for $\varphi \in \mathbb{L}^2(C\times C)\sim \mathbb{L}^2(C)\hat{\otimes}\mathbb{L}^2(C)$,

$$[\chi_{r_1\hat{\otimes}r_2}(\theta, X)\varphi](u_1, u_2) = e^{i<R_{-u_2}r_1+R_{-u_1}r_2,X>} \times\varphi(u_1+\theta, u_2+\theta). \tag{2.14}$$

Therefore, we have the following expression for the adjoint operator:

$$[\chi_{r_1\hat{\otimes}r_2}(\theta, X)^*\varphi](u_1, u_2) = e^{-i<R_{\theta-u_2}r_1+R_{\theta-u_1}r_2,X>} \times\varphi(u_1-\theta, u_2-\theta). \tag{2.15}$$

A very important point: we consider functions $f$on the group of motions that are functions of $X = (x, y)$ only (they do not depend on θ, i.e. they are the "trivial" lifts on the group M_2 of functions on the plane $\mathbb{R}^2$). For the Fourier transform, we get, with $\varphi \in \mathbb{L}^2(C)$, $r \in R$,

$$\begin{aligned}[\hat{f}(r)\varphi](u) &= \int_C \tilde{f}(R_{\theta-u}r)\varphi(u-\theta)d\theta \\ &= <\varphi(\theta), \overline{\tilde{f}}(R_{-\theta}r)>_{\mathbb{L}^2(\mathrm{C},\mathrm{d}\theta)},\end{aligned} \tag{2.16}$$

in which $\tilde{f}(V)$ denotes as before the usual Fourier transform over (the Abelian group) $\mathbb{R}^2$:

$$\tilde{f}(V) = \int_{\mathbb{R}^2} f(X)e^{-i<V,X>_{\mathbb{R}^2}} dxdy. \tag{2.17}$$

The adjoint of the Fourier transform is given by:

$$[\hat{f}(r)^*\varphi](u) = \overline{\tilde{f}}(R_{-u}r) < \varphi, 1 >_{\mathbb{L}^2(C)}, \tag{2.18}$$

where 1 is the constant function over C, with value 1.

It follows that:

$$\begin{aligned} [\hat{f}(r_1)^*\hat{\otimes}\hat{f}(r_2)^*\varphi](u_1, u_2) &= \overline{\tilde{f}}(R_{-u_2}r_1)\overline{\tilde{f}}(R_{-u_1}r_2) \\ &\times \int\int_{C\times C} \varphi(a,b)dadb \end{aligned} \tag{2.19}$$

The final formula we need, to compute the Generalized-Fourier-Descriptors, follows easily from Formula 2.14:

$$\begin{aligned} \hat{f}(r_1\hat{\otimes}r_2)\varphi(u_1, u_2) &= \int_C \tilde{f}(R_{\theta-u_2}r_1 + R_{\theta-u_1}r_2) \\ &\times\varphi(u_1-\theta, u_2-\theta)d\theta \end{aligned} \tag{2.20}$$

Using all these formulas, it is not hard to get the following formulas, for the Type-1 and Type-2 Generalized-Fourier-Descriptors:

$$[I_1^r(f)\varphi](u) = \int_C |\tilde{f}(R_\theta r)|^2 d\theta \ < \varphi, 1 >_{\mathbb{L}^2(C)}, \tag{2.21}$$

$$\begin{aligned} [I_2^{r_1,r_2}(f)\varphi](u_1, u_2) &= \int_C \tilde{f}(R_\theta(\hat{r}_1 + \hat{r}_2))\overline{\tilde{f}}(R_\theta\hat{r}_1) \\ &\quad \overline{\tilde{f}}(R_\theta\hat{r}_2)d\theta \int\int_{C\times C} \varphi(a,b)dadb, \\ \text{with } \hat{r}_i &= R_{-u_i}r_i, \qquad i = 1, 2. \end{aligned}$$

Clearly, these formulas are completely determined by the invariants used in the first part of the paper:

$$I_1^r(f) = \int_C |\tilde{f}(R_\theta r)|^2 d\theta, \ r \in R, \tag{2.22}$$

$$\begin{aligned} I_2^{\xi_1,\xi_2}(f) &= \int_C \tilde{f}(R_\theta(\xi_1+\xi_2))\overline{\tilde{f}}(R_\theta\xi_1)\overline{\tilde{f}}(R_\theta\xi_2)d\theta, \\ \text{for } \xi_1, \xi_2 &\in \mathbb{R}^2, \end{aligned}$$

We finish this section with two very important remarks:

Remark 4. *The Generalized-Fourier-Descriptors are real quantities (This is not an obvious fact for the second type invariants, but it is easily checked).*

Remark 5. *Let us define a one color (say grey-level) image f as a real compactly supported function over the plane $\mathbb{R}^2$, with support in a fixed compact region K of the plane (the "screen"). Then, the set of images is just $\mathbb{L}^2(K) \subset \mathbb{L}^2(\mathbb{R}^2)$. As it has been noticed in* [7], *if we set $f^\circ(x,y) = f(-x,y)$, The Fourier descriptors of f and f° are identical. As a consequence:*

-The Generalized Fourier Descriptors are not complete,

-They are not even weakly complete, in the sense that they do not discriminate in restriction to any residual subset of $\mathbb{L}^2(K)$*: no subset containing either* f *or* f° *(in the exclusive sense) can be residual.*

The purpose of the next sections of this paper is twofold:

1. We want to show that, nevertheless, the Generalized-Fourier-Descriptors are very natural, and not far being weakly complete.,

2. We want to exhibit a bigger set of invariants, which contains the (two types of) Generalized-Fourier-Descriptors, and which is actually weakly complete.

3. The case of compact (non-Abelian) groups

This is the most beautiful part of the theory, showing in a very convincing way that the formulas 2.13 are really pertinent: in the compact case, (including the classical Abelian case of exterior contours), the Generalized Fourier Descriptors are weakly complete. This is due to the beautiful old Tannaka-Krein duality theory. (See [14], [15]).

3.1. **Chu and Tannaka categories, Chu and Tannaka dualities.** Tannaka Theory is the generalization to compact groups of Pontriaguin's duality theory.

The following facts are standard: The dual of a compact group is a **discrete set**, and all its unitary irreducible representations are **finite dimensional**.

The main lines of Tannaka theory is like that: we are given a **separable** compact group G.

1. There is the notion of a Tannaka category $\mathcal{T}_G$, that describes the structure of the set of finite dimensional unitary representations of G;

2. There is the notion of a quasi representation $\mathcal{Q}$ of a Tannaka category $\mathcal{T}_G$;

3. The set $rep(G)\hat{}$ of quasi representations of the Tannaka category $\mathcal{T}_G$ has the structure of a topological group;

4. The groups $rep(G)\hat{}$ and G are naturally isomorphic. (Tannaka duality).

This scheme completely generalizes the scheme of Pontryagin's duality to the case of compact groups.

In fact, Tannaka duality theory is just a particular case of Chu duality, which will be **the crucial form of duality we need** for our practical purposes. Hence, let us introduce precisely Chu duality ([14], [5]) that we will need later, and Tannaka duality will just be **the particular case of compact groups**.

Let temporarily G be an arbitrary topological group.

For all $n \in \mathbb{N}$ the set $rep_n(G)$ denotes the set of continuous unitary representations of G over $\mathbb{C}^n$. $rep_n(G)$ is endowed with the following topology: a basis of open neighborhoods of $T \in rep_n(G)$ for this topology is given by the sets $W(K,T,\varepsilon), \varepsilon > 0$, and $K \subset G$, a compact subset,

$$W(K,T,\varepsilon) = \{\tau \in rep_n(G) \mid ||T(g) - \tau(g)|| < \varepsilon, \forall g \in K,$$

where the norm of operators is the usual Hilbert-Schmidt norm. If G is locally compact, so is $rep_n(G)$.

Definition 3. *The Chu-Category of G is the category $\pi(G)$, the objects of which are the finite dimensional unitary representations of G, and the morphisms are the intertwining operators.*

Definition 4. *A quasi-representation of the category $\pi(G)$ is a function Q over $ob(\pi(G))$ such that $Q(\chi)$ belongs to $U(H_\chi)$, the unitary group over the underlaying space H_χ of the representation χ, with the following properties:*

0. Q commutes with Hilbert direct-sum: $Q(\chi_1 \dot{\oplus} \chi_2) = Q(\chi_1) \dot{\oplus} Q(\chi_2)$

1. Q commutes with the Hilbert tensor product: $Q(\chi_1 \hat{\otimes} \chi_2) = Q(\chi_1) \hat{\otimes} Q(\chi_2)$,

2. Q commutes with the equivalence operators: for an equivalence A between χ_1 and χ_2, $A \circ Q(\chi_1) = Q(\chi_2) \circ A$,

3. the mappings, $rep_n(G) \to U(\mathbb{C}^n)$, $\chi \to Q(\chi)$ are continuous.

The set of quasi-representations of the category $\pi(G)$ is denoted by $rep(G)\hat{}$.

There are "natural" quasi representations of G : it is clear that, for each $g \in G$, the mapping $\Omega_g(\chi) = \chi(g)$ defines a quasi-representation of $\pi(G)$.

Remark 6. *$rep(G)\hat{}$ is a group with the multiplication $Q_1.Q_2(\chi) = Q_1(\chi).Q_2(\chi)$.*

The neutral element is E, with $E(\chi) = \Omega_e(\chi) = \chi(e)$, for e the neutral of G.

There is a topology over $rep(G)\hat{}$ such that it becomes a topological group. A fundamental system of neighborhoods of E is given by the sets $W(K_{n_1}^{\hat{}}, ..., K_{np}^{\hat{}}, \varepsilon)$, $\varepsilon > 0$ and $K_{n_i}^{\hat{}}$ is compact in $rep_{n_i}(G)$, with $W(K_{n_1}^{\hat{}}, ..., K_{np}^{\hat{}}, \varepsilon) = \{Q \in rep(G)\hat{} \mid ||Q(\chi) - E(\chi)|| < \varepsilon, \forall \chi \in \cup K_{n_i}^{\hat{}}\}$.

The first main result is that, as soon as G is locally compact, the mapping $\Omega : G \to rep(G)\hat{}$, $g \to \Omega_g$ is a continuous homomorphism.

Definition 5. *A locally compact G has the* ***duality property if*** *Ω is a topological group isomorphism.*

The main result is:

Theorem 4. *If G is locally compact, Abelian, then G has the duality property. (This is no more than Pontryagin's duality).*

If G is compact, G has the duality property. (This is Tannaka-Krein theory).

In the last section of the paper, for the purpose of pattern recognition, we will use crucially the fact that **another class of groups, namely the Moore groups, have also the duality property**.

3.2. **Generalized Fourier Descriptors over compact groups.** Our result in this section is based upon Tannaka theory, and shows that the **weak-completeness** (-i.e. completeness over a residual subset of $\mathbb{L}^2(G,dg)$-) of the Generalized-Fourier-Descriptors (which holds on the circle group, and which is crucial for pattern recognition of "exterior contours") **generalizes to compact separable groups**.

If G is compact separable, then, we have the following crucial but obvious lemma:

Lemma 1. *The subset R of functions $f \in \mathbb{L}^2(G,dg)$ such that $\hat{f}(\hat{g})$ is invertible for all $\chi = \hat{g} \in G^{\hat{}}$ is residual in $\mathbb{L}^2(G,dg)$.*

Proof. It follows from [6] that if G is compact separable, then $G^{\hat{}}$ is countable. For a fixed $\hat{g}$, the set of f such that $\hat{f}(\hat{g})$ is not invertible is clearly open, dense. Hence, R is a countable intersection of open-dense sets, in a Hilbert space. □

Now, let us take two functions $f, h \in R$, such that the associated Generalized-Fourier-Descriptors are equal. The equality of the first-type Fourier-Descriptors gives $\hat{f}(\hat{g}) \circ \hat{f}(\hat{g})^* = \hat{h}(\hat{g}) \circ \hat{h}(\hat{g})^*$, for all $\hat{g} \in G^{\hat{}}$. Since $\hat{f}(\hat{g})$ is invertible, we deduce that there is $u(\hat{g}) \in U(H_{\hat{g}})$, such that $\hat{f}(\hat{g}) = \hat{h}(\hat{g})\ u(\hat{g})$.

If χ is a reducible unitary representation, it is a finite direct sum of irreducible representations, and therefore, the equality $\hat{f}(\chi) \circ \hat{f}(\chi)^* = \hat{h}(\chi) \circ \hat{h}(\chi)^*$, for all $\hat{g}_i \in G^{\hat{}}$ also defines an invertible $u(\chi) = \hat{h}(\chi)^{-1}\ \hat{f}(\chi)$. (By the finite sum decomposition, $\hat{h}(\chi) = \dot{\oplus}\hat{h}(\hat{g}_i)$), hence $\hat{h}(\chi)$ is invertible.

Moreover, in this compact case, it is obvious that the mappings $rep_n(G) \to M(n, \mathbb{C})$, $\chi \to \hat{f}(\chi)$ are continuous and the mapping $\chi \to u(\chi) = \hat{h}(\chi)^{-1}\ \hat{f}(\chi)$ is also continuous.

Also, the equality of the (second) Fourier-Descriptors for the irreducible representations, due to the finite decomposition of any representation in a direct sum of irreducible ones, plus the usual properties of Hilbert tensor product shows that the equality of Fourier-Descriptors holds also for arbitrary (non-irreducible) unitary finite-dimensional representations, i.e., if χ, χ' are unitary representations, non necessarily irreducible, we have also:

$$\hat{f}(\chi)\hat{\otimes}\hat{f}(\chi') \circ \hat{f}(\chi\hat{\otimes}\chi')^* = \hat{h}(\chi)\hat{\otimes}\hat{h}(\chi') \circ \hat{h}(\chi\hat{\otimes}\chi')^*. \tag{3.1}$$

Replacing in this last equality $\hat{f}(\chi) = \hat{h}(\chi)\ u(\chi)$, and taking into account the fact that all the $\hat{f}(\chi), \hat{h}(\chi)$ are invertible, we get that:

$$u(\chi\hat{\otimes}\chi') = u(\chi)\hat{\otimes}u(\chi'), \tag{3.2}$$

for all finite dimensional unitary representations χ, χ' of G.

Now, for such χ, χ', and for A intertwining χ and χ', we have also $A\hat{f}(\chi) = \int_G f(g)A\chi(g^{-1})dg = \int_G f(g)\chi'(g^{-1})Adg = \hat{f}(\chi')A$. It follows that $A\hat{h}(\chi)u(\chi) = \hat{h}(\chi')u(\chi')A$, hence, $\hat{h}(\chi')Au(\chi) = \hat{h}(\chi')u(\chi')A$, in which $\hat{h}(\chi')$ is invertible. Therefore, $Au(\chi) = u(\chi')A$. By Definition 4, u is a quasi-representation of the category $\pi(G)$. By Theorem 4, G has the duality property, and for all $\hat{g} \in G^{\hat{}}$, $u(\hat{g}) = \chi_{\hat{g}}(g_0)$ for some $g_0 \in G$. Then:

$$\hat{f}(\hat{g}) = \hat{h}(\hat{g})\chi_{\hat{g}}(g_0),$$

and, by the main property 2.11 of Fourier transforms, $\hat{f} = \widehat{h_a}$, $f = h_a$ for some $a \in G$.

Therefore, we have proven the following main theorem.

Theorem 5. *Let G be a compact separable group. Let R be the subset of elements of $\mathbb{L}^2(G,dg)$ on which the Fourier transform takes value in invertible operators. Then R is residual in $\mathbb{L}^2(G,dg)$, and the Generalized Fourier Descriptors discriminate over R.*

4. The case of the group of motions with small rotations $M_{2,N}$

This section contains our final results. Some of them are rather strange: We already know (Remark 5) that Generalized Fourier descriptors do not discriminate over any residual subset of $\mathbb{L}^2(\mathbb{R}^2)$, with respect to the action of the group of motions M_2. Here, we will consider the action on the plane of the group $M_{2,N}$ of translations and small rotations. In the case where N is an odd number, we will be able to make a complete theory, and to get a weak-completion result.

4.1. **Moore groups and duality for Moore groups.** For details, we refer to [14]. We already know that compact groups have all their unitary irreducible representations of finite dimension. But they are not the only ones.

Definition 6. *A Moore group is a locally-compact group, such that all its unitary irreducible representations have finite-dimensional underlaying Hilbert space.*

Theorem 6. *The groups $M_{2,N}$ are Moore groups.*

Proof. These groups are semidirect products of the type $G_0 \ltimes \mathbb{R}^2$, where G_0 is a (Abelian) finite group. Then we can use Mackey's Imprimitivity Theorem to compute their dual (see [23] for instance). By this theorem, their unitary irreducible representations are parametrized by the (contragredient) action of the action of G_0 on $\mathbb{R}^2$, and their underlaying Hilbert spaces are the spaces of square summable functions on these orbits, with respect to the corresponding quasi-invariant measures. These orbits are finite. Hence, their $\mathbb{L}^2$-space is isomorphic to $\mathbb{C}^n$. □

Theorem 7. *(Chu duality) Moore groups (separable) have the duality property.*

Then, we will try to copy what has been done for compact groups to our Moore groups. There are several difficulties, due to the fact that the functions under consideration (the images) are very special functions over the group. In fact, they are functions over the homogeneous space $\mathbb{R}^2$ of $M_{2,N}$.

4.2. **Representations, Fourier transform and Generalized Fourier descriptors over $M_{2,N}$.** In fact, considering only "images", we will be interested only with functions on $M_{2,N}$, that are also functions on the plane $\mathbb{R}^2$. One of the main problems, as we shall see, is that there are several possible "lifts" of the functions of $\mathbb{L}^2(\mathbb{R}^2)$ on $\mathbb{L}^2(M_{2,N})$, and that the "trivial" lift is bad (in the sense that it does not lead to a set of complete invariants under the action of motions).

Typical elements of $M_{2,N}$ are still denoted by $g = (\theta, x, y) = (\theta, X)$, $X = (x, y) \in \mathbb{R}^2$, but now, $\theta \in \check{N} = \{0, ..., N-1\}$. Each such θ represents a rotation of angle $\frac{2\theta\pi}{N}$, that we still denote by R_θ.

The Haar measure is the tensor product of the uniform measure over $\check{N}$ and the Lebesgue measure over $\mathbb{R}^2$. The dual space $\hat{G}$ is the union of the discrete set $\mathbb{Z}$

(characters) with the "Slice of Cake" $\mathcal{S}$, corresponding to nonzero values of $r \in \mathbb{R}^2$ of angle $\alpha(r)$, $0 \leq \alpha(r) < \frac{2\pi}{N}$. The support of the Plancherel Measure is $\mathcal{S}$ (characters are of no interest).

The representations and the Fourier transform have similar expressions to the case of M_2. The only difference is that the orbits of the contragredient action of the rotations on the plane, that are circles in the M_2 case become the finite set $\check{N}$ in the $M_{2,N}$ case. Therefore, $\mathbb{L}^2(C,d\theta)$ is changed for $l^2(\check{N}) \approx \mathbb{C}^N$.

Here $\varphi \in \mathbb{C}^N$, i.e. $\varphi : \check{N} \to \mathbb{C}$. We have exactly the same formula as for M_2 :

$$[\chi_r(\theta, X).\varphi](u) = e^{i<r,R_u X>}\varphi(u+\theta), \tag{4.1}$$

but $r \in \mathcal{S}$ and the map $l^2(\check{N}) \to l^2(\check{N})$, $\varphi(u) \to \varphi(u+\theta)$, is just the $\theta-$shift operator over $\mathbb{C}^{\mathrm{N}}$.

The Fourier transform has a similar expression to formula 2.9:

$$\begin{aligned} [\hat{f}(r).\varphi](u) &= \sum_{\check{N}}(\int\int_{\mathbb{R}^2} f(\theta,x,y)e^{-i<r,R_{u-\theta}X>} \\ &\quad \times\varphi(u-\theta)dxdy) \end{aligned} \tag{4.2}$$

Similar computations to those of Section 2.3 lead to the final formula for the Fourier descriptors relative to the trivial lift of functions f over $\mathbb{R}^2$ into functions over $M_{2,N}$ (not depending on θ) :

$$I_1^r(f) = \sum_{\theta\in\check{N}} |\tilde{f}(R_\theta r)|^2 d\theta, \ r \in R, \tag{4.3}$$

$$\begin{aligned} I_2^{\xi_1,\xi_2}(f) &= \sum_{\theta\in\check{N}} \tilde{f}(R_\theta(\xi_1+\xi_2))\overline{\tilde{f}}(R_\theta\xi_1)\overline{\tilde{f}}(R_\theta\xi_2)d\theta, \\ \text{for } \xi_1,\xi_2 &\in \mathbb{R}^2. \end{aligned}$$

By our Theorem 3, these **Generalized Fourier Descriptors are still invariant under the action of $M_{2,N}$ on $\mathbb{L}^2(\mathbb{R}^2)$.**

Another point is very important: the remark 5 of Section 2.3 **still holds, but only if N is an even number**. This fact will be very important in the following. The reason is that, if N is odd, the change $f(x,y) \to f(-x,y)$ does not map functions on a given orbit to a function on the same orbit (of the contragredient action of $\check{N}$ over $\mathbb{R}^2$).

To finish this section, let us explain the main problem that appears when we try to generalize the theorem 5 of Section 3.2.

For this, we have to consider the special expression of the Fourier transform of the "trivial lift" of a function on the plane. We have a similar expression as in

Section 2.3, (formula 2.16):

$$[\hat{f}(r)\varphi](u) = \sum_{\check{N}} \tilde{f}(R_{\theta-u}r)\varphi(u-\theta) = <\varphi(\theta), \overline{\tilde{f}}(R_{-\theta}r)>_{l^2(\check{N})}. \tag{4.4}$$

The **crucial** point in the proof of the main theorem 5 is that the operators $\hat{f}(r)$ are all invertible. But, here, it is not at all the case: the operators defined by the formula above are far from invertible: **they always have rank 1**, as is seen on the formula.

To overcome this difficulty, we have to chose another lift of functions on the plane to functions on $M_{2,N}$. This is what we do in the next section.

4.3. **The cyclic-lift from $\mathbb{L}^2(\mathbb{R}^2)$ to $\mathbb{L}^2(M_{2,N})$.** From now on, we consider functions on $\mathbb{R}^2$, that are square-summable, and that have their support contained in a translated of a given compact set K (the "screen").

Given a compactly supported function in $\mathbb{L}^2(\mathbb{R}^2,\mathbb{R})$, we can define its average and its centroid, as follows:

$$av(f) = \int_K f(x,y)dxdy,$$
$$centr(f) = (x_f, y_f) = X_f = (\int_K xf(x,y)dxdy, \int_K yf(x,y)dxdy).$$

Definition 7. *The cyclic-lift of a compactly supported $f \in \mathbb{L}^2(\mathbb{R}^2,\mathbb{R})$, with nonzero average, onto $\mathbb{L}^2(M_{2,N})$ is the function $f^c(\theta,x,y) = f(R_\theta X + \frac{centr(f)}{av(f)})$.*

The set of K-supported functions, with zero centroid is a closed subspace of $\mathbb{L}^2(K)$. Hence it is a Hilbert-subspace, denoted by $\mathcal{H}$. The set $\mathcal{I}$ of elements of $\mathcal{H}$ with nonzero average is an open subset of $\mathcal{H}$, therefore it has the structure of a Hilbert manifold. This is important since we shall apply to this space the parametric transversality theorem of [1].

Definition 8. *from now on, a (grey level, or one-color) "image" f is a function such that f^c is well defined and $f^c(0,X)$ belongs to $\mathcal{I}$.*

Notice that moreover, usual images have positive value. (grey or color levels vary between zero and 1). This will be of no importance here in.

By the lemma 12 in appendix 3.4.7, we know that f and g differ from a motion angle $\frac{4k\pi}{N}$ if and only if f^c and g^c differ from a motion with angle equal to $\frac{2k\pi}{N}$.

In this way, we reduce the problem, of equivalence with rotation of certain multiples of a small angle to **the problem of equivalence of the cyclic lifts over** $M_{2,N}$.

This is the problem that we will treat now, with the same method as the one of Section 3 (case of compact groups). For crucial reasons that will appear clearly below, we will consider only the case of an **odd** $N = 2n+1$. We have already a good reason for this from section 4.2: the remark 5 of Section 2.3 still holds and also now, if N is odd, when k varies in $\check{N}$, $2k \operatorname{mod} N$ also varies in $\check{N}$.

4.4. **Fourier-transform, Generalized-Fourier-Descriptors of cyclic-lifts over $M_{2,2n+1}$.** Using the expression 4.1 of the unitary irreducible representations over $M_{2,N}$, easy computations give the following results:

For $r_1, r_2 \in \mathcal{S}$,

$$[\chi_{r_1 \hat{\otimes} r_2}(\theta, V)\varphi](u1, u2) = e^{i<R_{-u_2}r_1 + R_{-u_1}r_2, V>} \times\varphi(u_1+\theta, u_2+\theta) \tag{4.5}$$

Notice that this expression is exactly the same as 2.14. As a consequence, again:

$$\chi_{r_1 \hat{\otimes} r_2}(\theta, X)^*\varphi](u_1, u_2) = e^{-i<R_{\theta-u_2}r_1 + R_{\theta-u_1}r_2, X>} \times\varphi(u_1-\theta, u_2-\theta). \tag{4.6}$$

For the Fourier transform of a cyclic lift f^c, we get:

$$\begin{aligned}[\widehat{f^c}(r)\Psi](u) &= \\ &= \sum_{\alpha} \tilde{f}(R_{2\alpha+u}r)e^{i<R_{2\alpha+u}r, \frac{1}{av(f)}Xf>}\Psi(-\alpha), \\ &= \sum_{\alpha\in\check{N}} \tilde{f}(R_{u-2\alpha}r)e^{i<R_{u-2\alpha}r, \frac{1}{av(f)}Xf>}\Psi(\alpha)\end{aligned} \tag{4.7}$$

Here, as above, $\tilde{f}(V)$ denotes the usual 2-D Fourier transform of f at V. We get also:

$$[\widehat{f^c}(r)^*\Psi](u) = \sum_{\alpha\in\check{N}} \overline{\tilde{f}}(R_{\alpha-2u}r)e^{-i<R_{\alpha-2u}r, \frac{1}{av(f)}Xf>}\Psi(\alpha). \tag{4.8}$$

The last expression we need is:

$$\begin{aligned}[\widehat{f^c}(r_1\hat{\otimes}r_2)\varphi](u_1, u_2) &= \\ \sum_{\alpha\in\check{N}} &\tilde{f}(R_{2\alpha-u_2}r_1 + R_{2\alpha-u_1}r_2) \\ &e^{i<R_{2\alpha-u_2}r_1 + R_{2\alpha-u_1}r_2, \frac{1}{av(f)}Xf>}\varphi(u_1-\alpha, u_2-\alpha).\end{aligned} \tag{4.9}$$

Formula 4.8 leads to:

$$\begin{aligned}[\widehat{f^c}(r_1)^*\hat{\otimes}\widehat{f^c}(r_2)^* \circ \varphi(u_1, u_2) &= \\ \sum_{(\alpha_1,\alpha_2)\in\check{N}\times\check{N}} &\overline{\tilde{f}}(R_{\alpha_2-2u_2}r_1)\overline{\tilde{f}}(R_{\alpha_1-2u_1}r_2) \\ &e^{-i<R_{\alpha_2-2u_2}r_1 + R_{\alpha_1-2u_1}r_2, \frac{1}{av(f)}Xf>} \times \varphi(\alpha_1, \alpha_2).\end{aligned} \tag{4.10}$$

Now, we can perform the computation of the Generalized Fourier Descriptors. After computations based upon the formulas just established, we get for the self adjoint matrix $I_1^r(f) = \hat{f}(r) \circ \hat{f}(r)^*$:

$$I_1^r(f)_{l,k} = \sum_{j\in\check{N}} \tilde{f}(R_{l-2j}r)\overline{\tilde{f}}(R_{k-2j}r)e^{i<(R_l-R_k)R_{-2j}r, \frac{1}{av(f)}Xf>},$$

and for the phase invariants $I_2^{r_1,r_2}(f)$:

$$[I_2^{r_1,r_2}(f)\Psi](u_1,u_2) =$$

$$\sum_{j\in\check{N}}\sum_{(\omega_1,\omega_2)\in\check{N}} \tilde{f}(R_{2j-u_2}r_1 + R_{2j-u_1}r_2)\overline{\tilde{f}}(R_{\omega_2-2u_2+2j}r_1) \overline{\tilde{f}}(R_{\omega_1-2u_1+2j}r_2) \times e^{i<(I-R_{\omega 2}-u_2)R_{2j-u_2}r_1+(I-R_{\omega 1}-u_1)R_{2j-u_1}r_2,\frac{1}{av(f)}Xf>} \times \Psi(u_1,u_2).$$

Since N is odd, setting $m = 2j$, we get:

$$I_1^r(f)_{l,k} = \sum_{m\in\check{N}} \tilde{f}(R_{l-m}r)\overline{\tilde{f}}(R_{k-m}r)e^{i<(R_l-R_k)R_{-m}r,\frac{1}{av(f)}Xf>}, \tag{4.11}$$

and also, we see easily that $I_2^{r_1,r_2}(f)$ is completely determined by the quantities:

$$\widetilde{I_2^{r_1,r_2}}(f)(u_1,u_2,\omega_1,\omega_2) =$$

$$\sum_{m\in\check{N}} \tilde{f}(R_{m-u_2}r_1 + R_{m-u_1}r_2)\overline{\tilde{f}}(R_{\omega_2-2u_2+m}r_1) \times \overline{\tilde{f}}(R_{\omega_1-2u_1+m}r_2) \times e^{i<(I-R_{\omega 2}-u_2)R_{m-u_2}r_1+(I-R_{\omega 1}-u_1)R_{m-u_1}r_2,\frac{1}{av(f)}Xf>} \tag{4.12}$$

Setting $u_2 = -l_2, \omega_2 - 2u_2 = k2$, $u_1 = -l_1, \omega_1 - 2u_1 = k_1$, we get:

$$\widetilde{I_2^{r_1,r_2}}(f)(l_1,l_2,k_1,k_2) =$$

$$\sum_{m\in\check{N}} \tilde{f}(R_{m+l_2}r_1 + R_{m+l_1}r_2)\overline{\tilde{f}}(R_{k_2+m}r_1) \overline{\tilde{f}}(R_{k_1+m}r_2) \times e^{i<(R_{l_2}-R_{k_2})R_m r_1+(R_{l_1}-R_{k_1})R_m r_2,\frac{1}{av(f)}Xf>} \tag{4.13}$$

Remark 7. *Consider the particular case $l_2 = k_2, l_1 = k_1$, and set $\xi_1 = R_{k_2}r_1$, $\xi_2 = R_{k_1}r_2$, then, we get:*

$$\widetilde{I_2^{\xi_1,\xi_2}}(f)(l_1,l_2) = \sum_{m\in\check{N}} \tilde{f}(R_m(\xi_1+\xi_2))\overline{\tilde{f}}(R_m\xi_1)\overline{\tilde{f}}(R_m\xi_2). \tag{4.14}$$

Note that this is just the **discrete version of the (continuous) invariants of type 2**, in Formula 2.22. Note also that, making the change of variables $\xi_1 = R_{k_2}r_1$, $\xi_2 = R_{k_1}r_2$, $\xi_3 = R_{l_2}r_1 + R_{l_1}r_2$, we get:

$$\widetilde{I_3^{\xi_1,\xi_2,\xi_3}}(f) = \sum_{m\in\check{N}} \tilde{f}(R_m\xi_3)\overline{\tilde{f}}(R_m\xi_1)\overline{\tilde{f}}(R_m\xi_2) e^{i<R_m(\xi_3-\xi_1-\xi_2),\frac{1}{av(f)}Xf>}.$$

which is the final (discrete form of our invariants).

Therefore, at the end, we have 3 sets of Generalized-Fourier-Descriptors (type-1, type-2, type-3):

$$\begin{aligned}
I_1^r(f)_{l,k},\ \widetilde{I_2^{\xi_1,\xi_2}}(f) &\subset \widetilde{I_3^{\xi_1,\xi_2,\xi_3}}(f)\\
I_1^r(f)_{l,k} &= \sum_{m\in\check{N}} \tilde{f}(R_{l-m}r)\overline{\tilde{f}}(R_{k-m}r)\\
&\quad e^{i<(R_l-R_k)R_{-m}r,\frac{1}{av(f)}Xf>},\\
\widetilde{I_2^{\xi_1,\xi_2}}(f) &= \sum_{m\in\check{N}} \tilde{f}(R_m(\xi_1+\xi_2))\\
&\quad \overline{\tilde{f}}(R_m\xi_1)\overline{\tilde{f}}(R_m\xi_2),\\
\widetilde{I_3^{\xi_1,\xi_2,\xi_3}}(f) &= \sum_{m\in\check{N}} \tilde{f}(R_m\xi_3)\overline{\tilde{f}}(R_m\xi_1)\overline{\tilde{f}}(R_m\xi_2)\\
&\quad e^{i<R_m(\xi_3-\xi_1-\xi_2),\frac{1}{av(f)}Xf>}.
\end{aligned}$$

As we shall see, there are several good reasons, contrarily to the case of the descriptors 2.22 over M_2 (we know that they are not weakly complete, already), that these one are weakly complete (i.e. they discriminate on a residual subset of the set of images under the action of motions of angle $\frac{4k\pi}{2n+1}$, i.e. $\frac{2k'\pi}{2n+1}$).

4.5. **Completeness of the Discrete Generalized Fourier Descriptors .** This is a rather hard work. We try to follow the scheme of the proof of Theorem 5, and at some point, there is a crucial obstruction.

Here, as above, a compact $K\subset\mathbb{R}^2$ is fixed, containing a neighborhood of the origin (K is the "screen"), and an image is an element of $\mathcal{I}$, from Definition 8.

Let us consider the subset $\mathcal{G}\subset\mathcal{I}$ of "generic images", defined as follows. For $f\in\mathcal{I}$, as above, $\tilde{f}^c$ denotes the regular 2-D Fourier transform of $f^c(0,X)$ as an element of $\mathbb{L}^2(\mathbb{R}^2)$. Denote as above $X=(x,y)\in\mathbb{R}^2$ (but here X should be understood as a point of the frequency plane). The function $\tilde{f}^c(X)$ is a complex-valued function of X, analytic in X. (Paley-Wiener). For $r\in\mathbb{R}^2$, denote by $\omega_r\in\mathbb{C}^{\mathrm{N}}$ the vector $\omega_r=(\tilde{f}^c(R_0r,...,\tilde{f}^c(R_{\theta_i}r),...,\tilde{f}^c((R_{\theta_{N-1}}r))$.

Denote also by Ω_r the circulant matrix associated to ω_r. If F_N denotes the usual DFT matrix of order N (i.e. the $N\times N$ unitary matrix representing the Fourier Transform over the Abelian group $\mathbb{Z}/\mathbb{N}\mathbb{Z}$), then the vector of eigenvalues δ_r of Ω_r meets $\delta_r=F_N\omega_r$.

Definition 9. *The generic set $\mathcal{G}$ is the subset of $\mathcal{I}$ of elements such that Ω_r is an invertible matrix for all $r\in\mathbb{R}^2$, except for a (may be countable) set of isolated values of r, for which Ω_r has a zero eigenvalue with simple multiplicity.*

The next Lemma shows that if N is an odd integer number, then $\mathcal{G}$ is very big.

Lemma 2. *Assume that N is odd. Then, $\mathcal{G}$ is residual.*

Proof. We consider the following mappings $\varrho_k : \mathcal{I}\times\mathbb{R}^2 \to \mathbb{R}^2$, $k \in \check{N}$, $\varrho_k(f,r)$ is the (real and imaginary part of the) k^{th} eigenvalue of Ω_r (it makes sense to talk about the k^{th} eigenvalue since all circulant matrices are simultaneously diagonalized by the DFT F_N). Lemma 35 from the appendix shows that, applying Abraham's parametric transversality Theorem ([1]) to ϱ_k, we find a residual subset $\mathcal{G}_\mathrm{k} \subset \mathcal{I}$, such that $\varrho_k(f)$ is transversal to zero, for all $f \in \mathcal{G}_\mathrm{k}$. Here, $\varrho_k(f)(x)$ means $\varrho_k(f,x)$. Set $\mathcal{G} = \cap_{\mathrm{k}\in\check{N}}\mathcal{G}_\mathrm{k}$. Clearly, $\mathcal{G}$ is residual, and for $f \in \mathcal{G}$ (for dimension 2 and codimension 2 reasons) Ω_r can have a zero eigenvalue at isolated points of $\mathbb{R}^2$ only. A similar argument shows that at these special points the zero eigenvalue is simple. □

Remark 8. *Notice that here, once more, the fact that N is odd is crucial.*

Now let us take $f, g \in \mathcal{G}$, and assume that their discrete Generalized Fourier descriptors from Section 4.4 are equal.

We can apply the reasoning of Section 3.2, to construct a quasi-representation of the category $\pi(M_{2,N})$ at points where $\Omega_r(f)$ and $\Omega_r(g)$ are invertible only. Again here, we leave the reader to check that character representations play little role, and we care mostly about the other representations.

Recall the formula 4.7 for our Fourier Transform in the case of $\mathrm{M}_{2,N}$:

$$\begin{aligned} & \widehat{[f^c}(r)\Psi](u) = \\ &= \sum_{\alpha\in\check{N}} \tilde{f}(R_{u-2\alpha}r)e^{i<R_{u-2\alpha}r,\frac{1}{av(f)}Xf>}\Psi(\alpha) \\ &= \sum_{\alpha\in\check{N}} \tilde{f}^t(R_{u-2\alpha}r)\Psi(\alpha), \\ \text{with } f^t(x) &= f(x+\frac{X_f}{av(f)}) = f^c(0,x), \end{aligned}$$

by the basic property of the usual 2D Fourier transform with respect to translations.

Since N is odd (a crucial point again), it is also equal to:

$$[\hat{f}^c(r)\Psi](u) = \sum_{\alpha\in\check{N}} \tilde{f}^c(R_{u-\alpha}r)(C\Psi)(\alpha).$$

where C is a certain universal unitary operator.

This formula can also be read, in a matrix setting, as:

$$\hat{f}^c(r) = \Omega_r(f)C, \tag{4.15}$$

for a certain universal permutation matrix C.

Also, it is easy to see that, by the equality of the invariants, that the points where $\Omega_r(f)$ and $\Omega_r(g)$ are non-invertible are the same.

Out of these isolated points, we can apply the same reasoning as in the compact case, Section 3.2. Hence, the equality of the first invariants give:

$$\begin{aligned} \hat{f}^c(r)\hat{f}^c(r)^* &= \Omega_r(f)\Omega_r(f)^* = \\ \hat{g}^c(r)\hat{g}^c(r)^* &= \Omega_r(g)\Omega_r(g)^*. \end{aligned}$$

Since at nonsingular points $\Omega_r(f)$ and $\Omega_r(g)$ are invertible, this implies that there is a unitary matrix $U(r)$ such that $\hat{g}^c(r) = \hat{f}^c(r)U(r)$.

Let $I = \{r_i | \Omega_{r_i} \text{ is singular}\}$. Out of I, $U(r)$ is an analytic function of r, since $U(r) = [\hat{f}^c(r)]^{-1}\hat{g}^c(r)$.

Now, we will need some results about unitary representations, namely:

R1. Two finite dimensional unitary representations that are equivalent are unitarily equivalent,

and the more difficult one, that we state in our case only, and which is a consequence of the "Induction-reduction" Theorem of Barut [2]. (However, once one knows the result, he can easily check it by direct computations in the special case):

R2. For $r_1, r_2 \in \mathbb{R}^2$, the representation $\chi_{r_1 \hat{\otimes} r_2}$ is equivalent (hence unitarily equivalent by **R1**) to the direct Hilbert sum of representations $\dot{\oplus}_{k\in\check{N}}\chi_{r_1+R_kr_2}$.

This means that , if we take r_1, r_2 out of I, but $r_1 + R_{k_0}r_2 \in I$, and $r_1 + R_kr_2 \notin I$ for $k \neq k_0$ (which is clearly possible), then, if A denotes the unitary equivalence between $\chi_{r_1\hat{\otimes}r_2}$ and $\dot{\oplus}_{k\in\check{N}}\chi_{r_1+R_kr_2}$, setting $\xi_k = r_1 + R_kr_2$, we can write that the block diagonal matrix $\Delta_f = diag(\hat{f}^c(\xi_0), ..., \hat{f}^c(\xi_{N-1}))$ satisfies:

$$\Delta_f = \Delta_g AU(r_1)^*\hat{\otimes}U(r_2)^*A^{-1}. \tag{4.16}$$

Indeed, this comes from the equality of the second-type descriptors:

$$\hat{f}^c(\chi_{r_1})\hat{\otimes}\hat{f}^c(\chi_{r_2}) \circ \hat{f}^c(\chi_{r_1}\hat{\otimes}\chi_{r_2})^* = \tag{4.17}$$

$$\hat{g}^c(\chi_{r_1})\hat{\otimes}\hat{g}^c((\chi_{r_2}) \circ \hat{g}^c(\chi_{r_1}\hat{\otimes}\chi_{r_2})^*,$$

and since $\hat{g}^c(\chi_{r_1})\hat{\otimes}\hat{g}^c((\chi_{r_2}) = \hat{f}^c(\chi_{r_1})\hat{\otimes}\hat{f}^c(\chi_{r_2}) \circ U(r_1)\hat{\otimes}U(r_2)$ and both are invertible operators, then, replacing in 4.17, we get:

$$\hat{f}^c(\chi_{r_1}\hat{\otimes}\chi_{r_2}) \circ \hat{f}^c(\chi_{r_1})^*\hat{\otimes}\hat{f}^c(\chi_{r_2})^* =$$

$$\hat{g}^c(\chi_{r_1}\hat{\otimes}\chi_{r_2}) \circ U(r_1)^*\hat{\otimes}U(r_2)^* \circ \hat{f}^c(\chi_{r_1})^*\hat{\otimes}\hat{f}^c(\chi_{r_2})^*,$$

which implies,

$$\hat{f}^c(\chi_{r_1}\hat{\otimes}\chi_{r_2}) = \hat{g}^c(\chi_{r_1}\hat{\otimes}\chi_{r_2}) \circ U(r_1)^*\hat{\otimes}U(r_2)^*.$$

Using the equivalence A, we get:

$$A\hat{f}^c(\chi_{r_1}\hat{\otimes}\chi_{r_2})A^{-1} =$$

$$A\hat{g}^c(\chi_{r_1}\hat{\otimes}\chi_{r_2})A^{-1}A \circ U(r_1)^*\hat{\otimes}U(r_2)^*A^{-1}.$$

This last equality is exactly 4.16.

Remark 9. *The following fact is important: the matrix A is a constant. This comes again from the "Induction-Reduction" Theorem of* [2] *(or from direct computation): the equivalence* $A : \mathbb{L}^2(\check{N})\hat{\otimes}\mathbb{L}^2(\check{N}) \approx \mathbb{L}^2(\check{N} \times \check{N}) \rightarrow \dot{\oplus}_{k\in\check{N}}\mathbb{L}^2(\check{N})$*, is given by* $A\varphi = \dot{\oplus}_{k\in\check{N}}\varphi_k$*, with* $\varphi_k(l) = \varphi(l - k, l)$*. Hence, its matrix is independent of* r_1, r_2.

Let us rewrite 4.16 as $\Delta_f = \Delta_g H$, for some unitary matrix H. Since $N-1$ corresponding blocks in Δ_f and Δ_g are invertible, it follows that H is also block diagonal. Since it is unitary, all diagonal blocks are unitary. In particular, the k_0^{th} block is unitary. Also, $H = A \circ U(r_1)^* \hat{\otimes} U(r_2)^* A^{-1}$ is an analytic function of r_1, r_2. Moving r_1, r_2 in a neighborhood moves $r_1 + R_{k_0} r_2$ in a neighborhood. If we read the k_0^{th} line of the equality $\Delta_f = \Delta_g H$, we get $\Delta_f(\chi_{r_1+R_{k_0}r_2}) = \Delta g(\chi_{r_1+R_{k_0}r_2}) H_{k_0}(r_1, r_2)$, where $H_{k_0}(r_1, r_2)$ is unitary, and analytic in r_1, r_2. It follows that, by analyticity outside I, that $U(r)$ prolongs analytically to all of $\mathbb{R}^2$, in a unique way. The equality $\hat{g}^c(r) = \hat{f}^c(r)U(r)$ holds over $\mathbb{R}^2$.

Now, for the characters $\hat{K}_n$, $n \in \mathbb{Z}$, it is easily computed that $\hat{f}^c(\hat{K}_n) = av\,(f)$ $\sum_k e^{2\pi i n k/N}$. In particular $\hat{f}^c\,(0) = Nav\,(f)$.

The equality of the second type invariants imply that $av\,(f) = av\,(g)$; and hence, the equality of the Generalized Fourier descriptors relative to characters is implied by the equality of the others.

Moreover, if $\hat{f}^c(\hat{K}_n) \neq 0$, $\hat{g}^c(\hat{K}_n) = \hat{f}^c(\hat{K}_n)$, this suggests the choice $U(\hat{K}_n) = 1$ for all $n \in \mathbb{Z}$.

Now, let us define $U(\chi)$ for any p-dimensional representation χ (p arbitrary).

As a unitary representation χ is unitarily equivalent to $\overset{p}{\underset{i=1}{\dot{\oplus}}} \chi_{r_i} \overset{k}{\underset{i=1}{\dot{\oplus}}} \hat{K}_{n_i} = \dot{\oplus} \chi_i$, $r_i \in \mathcal{S}$,

i.e $\chi = A\Delta\chi_i A^*$, where A is some unitary matrix, and $\Delta\chi_i$ is a block diagonal of irreducible representations χ_i.

We define $U(\chi) = A\Delta U \chi_i A^*$.

Lemma 3. *U is well defined.*

Proof. Let A and B be such that:
$\chi = A\Delta\chi_i A^* = B\Delta\chi_i B^*$.
Then, $B^* A \Delta\chi_i = \Delta\chi_i B^* A$.
Consider a primary-labelling of $\Delta\chi_i$:
$\Delta\chi_i = \chi_1 \hat{\otimes} I_{d_{k_1}} \dot{\oplus} \ldots\ldots \dot{\oplus}\, \chi_p \hat{\otimes} I_{d_{k_p}}$, where $\chi_i \neq \chi_j$ for all $i \neq j$.

With an argument similar to the one at the end of the proof of Lemma 14 (from Formula 5.6 on), we get that:

$$B^* A = \left(Id_{k_1} \hat{\otimes} \Lambda_1\right) \dot{\oplus} \ldots\ldots\ldots \dot{\oplus} \left(Id_{k_p} \hat{\otimes} \Lambda_p\right),$$

where $\Lambda_1, \ldots\ldots\ldots, \Lambda_p$ are certain unitary matrices.

Then we have to show that
$B^* A \Delta U\,(\chi_i) = \Delta U\,(\chi_i)\, B^* A$, or equivalently :

$$B^* A \Delta U\,(\chi_i)\, AB^* = \Delta U\,(\chi_i) \tag{4.18}$$

This is true soon as:

$$U\,(\chi j)\,\hat{\otimes} Id_{k_j} = \left(Id_{k_j} \hat{\otimes} \Lambda_j\right)\left(U\,(\chi j)\,\hat{\otimes} Id_{k_j}\right)\left(Id_{k_j} \hat{\otimes} \Lambda_j\right)^*, \tag{4.19}$$

for all j.

But
$\left(Id_{k_j} \hat{\otimes} \Lambda_j\right)\left(U\,(\chi j)\,\hat{\otimes} Id_{k_j}\right)\left(Id_{k_j} \hat{\otimes} \Lambda_j\right)^* =$
$\left(Id_{k_j} \hat{\otimes} \Lambda_j\right)\left(U\,(\chi j)\,\hat{\otimes} Id_{k_j}\right)\left(Id_{k_j} \hat{\otimes} \Lambda_j^*\right) =$
$\left(U\left(\chi_j\right)\hat{\otimes} \Lambda_j\right)\left(Id_{k_j} \hat{\otimes} \Lambda_j^*\right) =$

$U\left(\chi_j\right)\hat{\otimes}\Lambda_j\Lambda_j^* = U\left(\chi_j\right)\hat{\otimes}Id_{k_j},$
Since Λ_j is unitary. This ends the proof. □

Lemma 4. *At a point*

$\chi = (A\chi_{r_1} \dot{\oplus} .... \dot{\oplus} \chi_{r_p} \dot{\oplus} \hat{K}_{k1} \dot{\oplus} .... \dot{\oplus} \hat{K}_{kl})A^*$
$= A(\chi_r \dot{\oplus} \chi_{\hat{K}})A^*$, *where* $r_{1,.......}r_p \notin I$
$U(\chi) = (A\hat{f}^c(\chi_r)^{-1}\hat{g}^c(\chi_r) \dot{\oplus} Id)A^*$

Proof. By definition of I at such points $r_{1,.......}r_p$, $\hat{f}^c(\chi_r)$ and $\hat{g}^c(\chi_r)$ are invertible. Also, by equality of the first descriptors, $\hat{f}^c\left(\chi_{rj}\right)\ \hat{f}^{c^*}\left(\chi_{r_j}\right) = \hat{g}^c\left(\chi_{rj}\right)\ \hat{g}^{c^*}\left(\chi_{r_j}\right)$, we have $\hat{g}^c\left(\chi_{rj}\right) = \hat{f}^c\left(\chi_{rj}\right)U\left(\chi_{rj}\right)$. Also, by definition, $U(\hat{K}_j) = Id$. This shows the result. □

Lemma 5. $U(\chi \dot{\oplus} \chi') = U(\chi) \dot{\oplus} U(\chi')$.

Lemma 6. *If* $A\chi = \chi' A$, *A unitary, then:* $AU(\chi) = U(\chi')A$.

The Lemmas 5, 6 are just trivial consequences of the definition of $U(\chi)$.

Lemma 7. *U is continuous.*

This is the most complicated point. We shall need crucially the Lemma (14) at the end of the appendix.

Proof. Assume that $\chi^p \in \operatorname{Re} p_n(G)\hat{}$, $\chi^p \longrightarrow \chi'$
set $\chi' = B(\chi_1\hat{\otimes}I_{k_1} \dot{\oplus} .... \dot{\oplus} \chi_p\hat{\otimes}I_{k_p})B^* = B\chi B^*$
with $\chi_i \neq \chi_j$ for $i \neq j$.
Then, we apply to $B^*\chi^p B$ the result of Lemma (14). $B^*\chi^p B$ tends to χ, iff $B^*\chi^p B$ meets the statements 1,2,3 of Lemma (14).

Using the notations of Lemma (14), it follows that $B^*\chi^p B = A_p(\dot{\oplus}_{i,j}\chi_{\varrho^p_{i,j}} \dot{\oplus}_{i',j'} \hat{K}_{n^p_{i,j}})A_p^*$, with properties 1.2.3.
By definition of U,

$$\begin{aligned} U(B^*\chi^p B) &= A_p(\dot{\oplus}_{i,j}U(\chi_{\varrho^p_{i,j}}) \dot{\oplus} U(\hat{K}_{n^p_{i,j}}))A_p^* \\ &= A_p(\dot{\oplus}_{i,j}U(\chi_{\varrho^p_{i,j}}) \dot{\oplus} Id)A_p^*, \end{aligned}$$

and, for any convergent subsequence A_p,
$A_p \longrightarrow A = (I_{k_1}\hat{\otimes}\Lambda_1 \dot{\oplus} .... \dot{\oplus} I_{k_p}\hat{\otimes}\Lambda_p)$ and using Lemma 6,

$$U(B^*\chi^p B) \longrightarrow A((U\left(\chi_{r_1}\right)\hat{\otimes}I_{k_1})\dot{\oplus} .... \dot{\oplus} (U\left(\chi_{rp}\right)\hat{\otimes}I_{k_p}))A^*$$

$B^*U(\chi^p)B \longrightarrow \dot{\oplus}_j\left(I_{kj}\hat{\otimes}\Lambda_j\right)\left(U\left(\chi_{r_j}\right)\hat{\otimes}I_{k_j}\right)\left(I_{kj}\hat{\otimes}\Lambda_j\right)^* \dot{\oplus} Id.$

Then, $B^*U(\chi^p)B \longrightarrow \dot{\oplus}_j\left(U\left(\chi_{r_j}\right)\hat{\otimes}\Lambda_j\right)(I_{kj}\hat{\otimes}\Lambda_j^*) \dot{\oplus} Id,$
$\longrightarrow \dot{\oplus}_j(U\left(\chi_{r_j}\right)\hat{\otimes}\Lambda_j\Lambda_j^*) \dot{\oplus} Id,$
$\longrightarrow \dot{\oplus}_j\left(U\left(\chi_{r_j}\right)\hat{\otimes}Id_{kj}\right) \dot{\oplus} Id.$

Therefore,

$$\begin{aligned} U(\chi^p) &\longrightarrow B(\dot{\oplus}_j \left(U\left(\chi_{r_j}\right)\hat{\otimes} Id_{kj}\right) + Id)B^* \\ &= BU(\chi)B^*. \end{aligned}$$

Hence by Lemma 6,

$$\begin{aligned} U(\chi^p) &\longrightarrow U(B\chi B^*) \\ &= U(\chi^{'}). \end{aligned}$$

Exhausting all convergent subsequences $A^p \longrightarrow A$ (not the same, may be) it remains only a finite number of terms and, for each corresponding subsequence $U(\chi^p) \longrightarrow U(\chi^{'})$.

Therefore the whole sequence $U(\chi^p)$ meets:

$U(\chi^p) \longrightarrow U(\chi^{'})$ and U is sequentially continuous hence continuous. □

Lemma 8. $U(\chi\hat{\otimes}\chi^{'}) = U(\chi)\hat{\otimes}U(\chi^{'})$

Proof. $\chi = A(\chi_1 \dot{\oplus} .... \dot{\oplus} \chi_l)A^* = A\Delta\chi A^*$.

$\chi^{'} = B(\chi_1^{'} \dot{\oplus} .... \dot{\oplus} \chi_p)B^* = B\Delta\chi^{'} B^*$.

$\chi\hat{\otimes}\chi^{'} = A(\chi_1 \dot{\oplus} .... \dot{\oplus} \chi_l)A^*\hat{\otimes}B(\chi_1^{'} \dot{\oplus} .... \dot{\oplus} \chi_p^{'})B^*$

$= \left(A\hat{\otimes}B\right)(\chi_1 \dot{\oplus} .... \dot{\oplus} \chi_l)\hat{\otimes}A^*(\chi_1^{'} \dot{\oplus} .... \dot{\oplus} \chi_p^{'})B^*$

$= \left(A\hat{\otimes}B\right)(\chi_1 \dot{\oplus} .... \dot{\oplus} \chi_l)\hat{\otimes}(\chi_1^{'} \dot{\oplus} .... \dot{\oplus} \chi_p^{'})\left(A^*\hat{\otimes}B^*\right)$

$= \left(A\hat{\otimes}B\right)(\chi_1 \dot{\oplus} .... \dot{\oplus} \chi_l)\hat{\otimes}(\chi_1^{'} \dot{\oplus} .... \dot{\oplus} \chi_p^{'})\left(A\hat{\otimes}B\right)^*$

$= \left(A\hat{\otimes}B\right)(\dot{\oplus}_{i,j}\chi_i\hat{\otimes}\chi_j^{'})\left(A\hat{\otimes}B\right)^*$.

$U(\chi\hat{\otimes}\chi^{'}) = \left(A\hat{\otimes}B\right)\dot{\oplus}_{i,j}\, U(\chi_i\hat{\otimes}\chi_j^{'})\left(A\hat{\otimes}B\right)^*$ (by Lemmas 5, 6).

Assume that:

$$U(\chi_i\hat{\otimes}\chi_j^{'}) = U(\chi_i)\hat{\otimes}U(\chi_j^{'}). \tag{4.20}$$

Then, $U(\chi\hat{\otimes}\chi^{'}) = \left(A\hat{\otimes}B\right)\dot{\oplus}_{i,j}\, U(\chi_i)\hat{\otimes}U(\chi_j^{'})\left(A\hat{\otimes}B\right)^*$

$$\begin{aligned} &= \left(A\hat{\otimes}B\right)U(\Delta\chi)\hat{\otimes}U(\Delta\chi^{'})\left(A\hat{\otimes}B\right)^* \\ &= \left(AU(\Delta\chi)\hat{\otimes}BU(\Delta\chi^{'})\right)\left(A^*\hat{\otimes}B^*\right) \\ &= AU(\Delta\chi)A^*\hat{\otimes}BU(\Delta\chi^{'})B^* \\ &= U(\chi)\hat{\otimes}U(\chi^{'}), \text{ by Lemma 6.} \end{aligned}$$

It remains to prove 4.20.

If χ_i and χ_j are both characters, then 4.20 can be rewritten as $1\hat{\otimes}1 = 1$.

If χ_i is not character and χ_j is, 4.20 can be rewritten as:

$$U(\chi_r\hat{\otimes}\hat{K}_n) = U(\chi_r)\hat{\otimes}U(\hat{K}_n). \tag{4.21}$$

It is easy to check that : $e^{in\ u}Id \circ \chi_r\varphi(u) = (\chi_r\hat{\otimes}\hat{K}_n) \circ e^{in\ u}Id\varphi(u)$.

Therefore by Lemma 6,

$$e^{in\ u}U(\chi_r) = U(\chi_r\hat{\otimes}\hat{K}_n)e^{in\ u},$$

or $U(\chi_r) = U(\chi_r \hat{\otimes} \hat{K}_n)$.
Since $U(\hat{K}_n) = 1$,
$U(\chi_r)\hat{\otimes}U(\hat{K}_n) = U(\chi_r)\hat{\otimes}1 = U(\chi_r) = U(\chi_r \hat{\otimes} \hat{K}_n)$.
The last case is to show: $U\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right) = U\left(\chi_{r_1}\right)\hat{\otimes}U\left(\chi_{r_2}\right).$

Actually, this is true if r_1, r_2 and $r_1 + R_k r_2 \notin I$ for all $k \in \breve{N}$: By the equality of the second Descriptors, $\hat{g}^c\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right) = \hat{f}^c\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right)U\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right)$, $\hat{g}^c\left(\chi_{r_1}\right)\hat{\otimes}\hat{g}^c\left(\chi_{r_2}\right) = \hat{f}^c\left(\chi_{r_1}\right)\hat{\otimes}\hat{f}^c\left(\chi_{r_2}\right)U\left(\chi_{r_1}\right)\hat{\otimes}U\left(\chi_{r_2}\right).$
Then,

$$\hat{g}^c\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right)\hat{g}^c\left(\chi_{r_1}\right)^*\hat{\otimes}\hat{g}^c\left(\chi_{r_2}\right)^* =$$
$$\hat{f}^c\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right)U\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right)U\left(\chi_{r_1}\right)^*\hat{\otimes}U\left(\chi_{r_2}\right)^*$$
$$\circ\hat{f}^c\left(\chi_{r_1}\right)^*\hat{\otimes}\hat{f}^c\left(\chi_{r_2}\right)^*$$
$$= \hat{f}^c\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right)\hat{f}^c\left(\chi_{r_1}\right)^*\hat{\otimes}\hat{f}^c\left(\chi_{r_2}\right)^*.$$

But, since $r_1, r_2, r_1+R_k r_2 \notin I$, $\hat{f}^c\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right)$ is invertible (remind that $\chi_{r_1}\hat{\otimes}\chi_{r_2} \approx \dot{\oplus}_k \chi_{(r_1+R_k r_2)}$).
Therefore, $U^*\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right)U\left(\chi_{r_1}\right)\hat{\otimes}U\left(\chi_{r_2}\right) = Id$,
$U\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right) = U\left(\chi_{r_1}\right)\hat{\otimes}U\left(\chi_{r_2}\right).$
But, the set of $(r_1, r_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that this holds is open, dense.
Otherwise, the mapping $(\chi, \chi') \longrightarrow \chi\hat{\otimes}\chi'$ is clearly continuous, and U is continuous by the Lemma 7. Also, the mapping $(r, \alpha, X) \longrightarrow \chi_r(\alpha, X)$ is continuous (it is analytic in (r, α, X)). Hence, on any compact $K \subset M_{2,N}$, the mapping $r \longrightarrow \chi_{r|k}$ is continuous. Therefore, in the diagram,

$$\begin{array}{ccccc} (r_1, r_2) & \rightarrow & \chi_{r_1}\hat{\otimes}\chi_{r_2} & \rightarrow & U(\chi_{r_1}\hat{\otimes}\chi_{r_2}) \\ \downarrow & & & & \downarrow \\ U(\chi_{r_1})\hat{\otimes}U(\chi_{r_2}) & & \rightarrow & & \begin{array}{c} U^*\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right)\circ \\ U\left(\chi_{r_1}\right)\hat{\otimes}U\left(\chi_{r_2}\right) \end{array} \end{array}$$

all arrows are continuous maps.
It follows that $U\left(\chi_{r_1}\hat{\otimes}\chi_{r_2}\right) = U\left(\chi_{r_1}\right)\hat{\otimes}U\left(\chi_{r_2}\right)$, since it is true on a dense subset of $\mathbb{R}^2 \times \mathbb{R}^2$. □

Lemmas 3, 4, 5, 6, 7, show that U is a quasi-representation of the category $\pi(M_{2,N})$.
Since $M_{2,N}$ has the duality property, $U(\chi) = \chi(g_0)$ for some $g_0 \in M_{2,N}$.
Also, we have:
$\hat{g}^c(\chi_r) = \hat{f}^c(\chi_r)U(\chi_r) = \hat{f}^c(\chi_r)\chi_r(g_0) = \hat{f}^c_{g_0}(\chi_r)$,
by the fundamental property of the Fourier transform.
The support of the Plancherel's measure being given by the (non-character) unitary irreducible representations χ_r, by the inverse Fourier transform, we get $g^c = f^c_{g_0}$, for some $g_0 \in M_{2,N}$, which is what we needed to prove. By lemma 12 we have shown our final result.

Theorem 8. *If the (Three types of) Discrete Generalized Fourier Descriptors of two images $f, g \in \mathcal{G}$ are the same, and if N is odd, then the two images differ from a translation, the rotation of which has angle $\frac{4k\pi}{N}$ (i.e. $\frac{2k'\pi}{N}$ since N is odd) for some k. Remind that $\mathcal{G}$ is a residual subset of the set of images of size K.*

4.6. **Discussion of the Theoretical results, and theoretical perspectives.** 1. We have to mention the final form of duality Theory, which is given by "Tatsuuma Duality", see [14], [19]. This is a generalization of Chu duality, to general locally compact type 1 groups. In particular, it works for M_2. Unfortunately, huge difficulties appear when using it in our context. However, it is already clear for us that, using this duality result, one could try to get a third type of invariants for M_2. This is a challenging, but hard subject.

2. The first and second-type Descriptors, that arise via the trivial or the cyclic lift have a very interesting practical feature: they don't depend on an estimation of the centroid of the image. This is a strong point in practice.

3. Otherwise, the variables that appear in the Generalized-Fourier-descriptors have clear frequency interpretation. Hence, depending on the problem (a high or low frequency texture), one can chose the values of these frequency variables in certain adequate ranges.

5. Appendix

We start with the statement of 3 very elementary lemmas, the proof of which is easy and left to the reader.

Lemma 9. *Let $\{a_n\}, \{b_n\}, n \in \mathbb{Z}$, be two sequences in $\mathbb{R}/2\pi\mathbb{Z}$ with $a_{-n} = -a_n$, $b_{-n} = -b_n$, and for all m, n,*

$$a_n + a_m - a_{m+n} = b_m + b_n - b_{n+m}, \tag{5.1}$$

then:

$$\begin{aligned} a_0 &= b_0 = 0, \\ \frac{a_n}{n} - \frac{a_m}{m} &= \frac{b_n}{n} - \frac{b_m}{m}, \text{ for all } m, n \geq 1. \end{aligned} \tag{5.2}$$

Conversely, 5.2 implies $a_n + a_m - a_{m+n} = b_m + b_n - b_{n+m}$.

Lemma 10. *Let f, g be real $\mathbb{L}^2$ functions on the circle. Let $\{f_n\}, \{g_n\}$ be their respective Fourier series.*

Assume that: a) $|f_n| = |g_n| \neq 0, \forall n \in \mathbb{Z}$, b) $f_n f_m \bar{f}_{n+m} = g_n g_m \bar{g}_{n+m}$ $\forall n, m \geq 1$. Then g is a translate of f.

Lemma 11. *The set of real $\mathbb{L}^2$ functions f on the circle, such that $f_n \neq 0$ for all $n \in \mathbb{Z}$ (where f_n is the Fourier series of f) is residual in $\mathbb{L}^2$.*

The fourth lemma below justifies the use of the "cyclic lift" of a function f over the plane to a function f^c over one of our motion groups M_2 or $M_{2,N}$.

Lemma 12. *Two functions $f, g \in L^2(\mathbb{R}^2)$ with nonzero average differ from a motion $(\theta, a, b) = (\theta, A)$ iff their cyclic lifts differs from a motion, the rotation component of which has angle $\frac{\theta}{2}$, and the translation is zero.*

Proof. Set $g(X) = f(R_\omega X + A)$ for $(\omega, A) \in G = M_2$ or $M_{2,N}$.

Then, $av(g) = \int_{\mathbb{R}^2} f(R_\omega X + A)dX = \int_{\mathbb{R}^2} f(R_\omega X + A)d(R_\omega X) = \int_{\mathbb{R}^2} f(Y)d(Y)$ $= av(f)$.

Also, $centr(g) = X_g = \int_{\mathbb{R}^2} Xf(R_\omega X+A)dX = R_{-\omega} \int_{\mathbb{R}^2} R_\omega X f(R_\omega X+A)d(R_\omega X) =$ $R_{-\omega} \int_{\mathbb{R}^2} (R_\omega X+A)f(R_\omega X+A)d(R_\omega X+A)) - R_{-\omega} A \int_{\mathbb{R}^2} f(R_\omega X+A)d(R_\omega X+A))$ $= R_{-\omega} X_f - R_{-\omega} A av(f)$. Hence we get two first conclusions:

$$\text{(5.3)} \qquad \begin{aligned} \text{For } g(X) &= f(R_\omega X + A), \\ 1.\ av(g) &= av(f), \\ 2.\ X_g = R_{-\omega}&(X_f - A av(f)). \end{aligned}$$

Now, consider the cyclic lifts f^c, g^c of f and g :

$$\begin{aligned} f^c(\alpha, X) &= f(R_\alpha X + \frac{1}{av(f)} X_f), \\ g^c(\alpha, X) &= g(R_\alpha X + \frac{1}{av(g)} X_g) \\ &= f(R_\omega(R_\alpha X + \frac{1}{av(f)} R_{-\omega}(X_f - A av(f))) + A) \\ &= f(R_{\omega+\alpha} X + A + \frac{1}{av(f)}(X_f - A av(f))) \\ &= f(R_{\omega+\alpha} X + \frac{1}{av(f)} X_f) \end{aligned}$$

Otherwise $(\lambda, B) f^c(\alpha, X) = f(R_{\alpha+\lambda}(R_\lambda X + B) + \frac{1}{av(f)} X_f) = f(R_{\alpha+2\lambda}\ X + R_{\alpha+\lambda} B + \frac{1}{av(f)}\ X_f)$. Therefore, choosing $\lambda = \frac{\omega}{2}$ and $B = 0$ we get:

$(\lambda, B) f^c(\alpha, X) = f(R_{\alpha+\omega} X + \frac{1}{av(f)} X_f) = g^c(\alpha, X)$.

Conversely, we assume that $(\lambda, 0) f^c(\alpha, X) = g^c(\alpha, X)$. This means that $f^c(\alpha + \lambda, R_\lambda X) = g^c(\alpha, X)$ which is equivalent to:

$$f(R_{\alpha+\lambda} R_\lambda X + \frac{1}{av(f)} X_f) = g(R_\alpha X + \frac{1}{av(g)} X_g).$$

This is true for all α, X. Let us take the particular case where $\alpha = -2\lambda$. It gives:

$$f(X + \frac{1}{av(f)} X_f) = g(R_{-2\lambda} X + \frac{1}{av(g)} X_g).$$

This is true for all X. Let us set $Y = X + \frac{1}{av(f)} X_f$. Then $X = Y - \frac{1}{av(f)} X_f$, and for all Y, we have:

$$f(Y) = g(R_{-2\lambda} Y + \frac{1}{av(g)} X_g - \frac{1}{av(f)} R_{-2\lambda} X_f).$$

$$f(Y) = g(R_{-2\lambda} Y + H),$$

for a certain H. This shows that f and g differ from a motion, with rotation angle 2λ. □

The following lemma is a more or less obvious technical result we need in Section 4.5. A compact $K \subset \mathbb{R}^2$ is fixed, containing a neighborhood of the origin. The set $\mathcal{H} \subset \mathbb{L}^2(K, \mathbb{R})$ is formed by the functions f that have there centroid $X_f = 0$. $\mathcal{H}$ is a closed subspace in a Hilbert space, hence it is a Hilbert subspace. The set $\mathcal{I}$ of images (of size K) is the open subset of $\mathcal{H}$ formed by the functions f with nonzero average. Let $N \in \mathbb{N}$ and $r \in \mathbb{R}^2$ be fixed, $r \neq 0$ Consider the map $\mathcal{M} : \mathcal{I} \rightarrow \mathbb{C}^N$, $f \rightarrow \omega_r = (\tilde{f}(R_0 r, ..., \tilde{f}(R_{\theta_i} r), ..., \tilde{f}((R_{\theta_{N-1}} r))$, where $\tilde{f}$ is the usual 2D Fourier transform of f as an element of $\mathbb{L}^2(\mathbb{R}^2, \mathbb{R})$.

Lemma 13. *$\mathcal{M}$ is a linear submersion if and only if N is odd.*

The proof is easy and left to the reader. A very simple idea for the proof is to show that, for suitably chosen $X_m \in K$, the distributions that are linear real combinations $f = \sum_m \alpha_m \delta_{X_m}$, where δ_{X_m} is the Dirac function at X_m, the $\mathcal{I}(f)$ span the realification of $\mathbb{C}^N$. Although, if N is even, this is clearly not true.

Now, we state and prove a lemma characterizing the convergence of sequences on $\text{rep}_n(M_{2,N})$. This Lemma is crucial to prove the continuity of the quasi-representation of $\pi(M_{2,N})$ that we construct in section 4.5.

Let χ^p be a sequence of finite dimensional representations of $M_{2,N}$ of the same dimension n. Assume that:

$$\chi = \chi_1 \otimes I_{k_1} \dot{\oplus} .... \dot{\oplus} \chi_p \otimes I_{kp}, \tag{5.4}$$

where χ_j is either a character $\chi_j = \hat{K}_{n_j}(\alpha, X) = e^{i n_j \alpha}$, or an irreducible representation of the form $\chi_{r_j}, r_j \in \mathcal{S}$, and $\begin{cases} r_i \neq r_j \\ n_i \neq n_j \end{cases}$ for $i \neq j$.

Let $\mathcal{S}_\varepsilon$ denote the "modified slice of cake", i.e. $\mathcal{S}_\varepsilon = \{(\lambda \cos \alpha, \lambda \sin \alpha), \lambda > 0, -\varepsilon \leq \alpha < \frac{2\pi}{N} - \varepsilon\}$. Chose ε small enough for $r_j \in \mathcal{S}_\varepsilon$ for all j.

Lemma 14. *$\chi^p \longrightarrow \chi$ if and only if there exists A_p, a unitary matrix, and $\varrho^p_{i,j}$, $n^p_{i,j}$ such that:*

1. $\begin{cases} \varrho^p_{i,j} \longrightarrow r_i \in \mathcal{S}_\varepsilon, \\ \hat{K}^p_{i,j} \longrightarrow \hat{K}_i, \end{cases}$
2. $\chi^p = A_p(\dot{\oplus}_{i,j} \chi_{\varrho^p_{i,j}} \dot{\oplus}_{i',j'} \hat{K}^p_{n_{i',j'}})A^*_p$
3. *For all convergent subsequence $A^p \longrightarrow A$,*

$A = I_{k_1} \hat{\otimes} \Lambda_1 \dot{\oplus} .... \dot{\oplus} I_{k_p} \hat{\otimes} \Lambda_p$, *for certain unitary matrices* $\Lambda_1, ..., \Lambda_p$.

Proof. : χ^p is *completely* reducible. Then:

$$\chi^p = A_p \Delta \chi_p A^*_p,$$

where $\Delta\chi^p$ is a block-diagonal of irreducible representations (either $\chi_{r^p_j}$ or $\hat{K}_{n^p_j}$). First, when $p \longrightarrow +\infty$, all the r^p_j and n^p_j remain bounded : Both would contradict the equicontinuity on any compact $K \subset M_{2,N}$ of the sequence $\chi^p_{|K}$ (χ^p restricted to K). Second, consider any convergent subsequence (still denoted by A_p) and the corresponding subsequences (r^p), (n^p). Note that the vectors (r^p) and (n^p) may have different dimensions depending on p.

In the following we shall consider extracted subsequences such that $(r^p_j), (\hat{K}^p_j)$ both converge. We shall show that all of them converge to the same required limit. Hence (after some multiplication of A_p by a constant matrix) the whole extracted sequence A_p will converge to a limit with the required form.

Since $(\hat{K}^p_j)$ converges, and since $(\hat{K}^p_j)$ is bounded among characters, $\hat{K}^p_j$ is constant, after a certain rank, $\hat{K}^p_j = \hat{K}^*_j$, and also, $\varrho^p_j \longrightarrow \varrho^*_j$.

The corresponding diagonal matrix we denote by $\Delta\chi^{'}$. We have $\chi_p - \chi = (A_p - A)\,\Delta\chi_p A_p^* + A\Delta\chi_p \left(A_p^* - A^*\right) + A\Delta\chi_p A^* - \chi$.

This shows that $A\Delta\chi_p A^* - \chi \longrightarrow 0$ (since $(A_p - A) \longrightarrow 0$ and since all other terms remain bounded in restriction to any compact $K \subset M_{2,N}$). Now, $\Delta\chi_p \longrightarrow \Delta\chi^{'}$. Hence $A\Delta\chi_p A^* - \chi = A\left(\Delta\chi_p - \Delta\chi'\right)A^* + A\Delta\chi' A^* - \chi$. It follows that $A\Delta\chi^{'} A^* - \chi = 0$ ($\Delta\chi_p$ converges uniformly to $\Delta\chi^{'}$ on any compact $K \subset M_{2,N}$).

$$A\Delta\chi^{'} = \chi A. \tag{5.5}$$

The representations $\Delta\chi^{'}$ and $\Delta\chi$ are equivalent. This shows that $\hat{K}_j^* = \hat{K}_j$, $\varrho_j^* = r_j$, with adequate multiplicity. We can find a unitary transformation P of $\mathbb{C}^{\mathrm{n}}$ such that $P\Delta\chi^{'} P^* = \Delta\chi$.

Then we change A for AP^*, A_p for $A_p P^*$ and $\Delta\chi_p$ for $P\Delta\chi_p P^*$. We have now: $A\Delta\chi = \chi A$, and $A = A_1 \dot{\oplus}$ ....$\dot{\oplus} A_p$.

Let us consider a non-character-block of this decomposition, the first block A_1 say.

The relation $A\Delta\chi = \chi A$ gives (considering the block decomposition of A_1 in $N \times N$ dimensional blocks) $A_1 = (A_{1i,j})$:

$$A_{1i,j}\chi_{r_1} = \chi_{r_1} A_{1i,j}. \tag{5.6}$$

By Shur's Lemma, $A_{1i,j}$ is a scalar multiple of the identity.

$A_{1i,j} = \lambda_{ij} Id$. This can be rewritten as :

$A_1 = I_{k_1} \hat{\otimes} \Lambda_1$, $A_1\left(\chi_{r_1} \hat{\otimes} I_{k_1}\right) = \left(\chi_{r_1} \hat{\otimes} I_{k_1}\right) A_1$.

It follows since A_1 is unitary that Λ_1 is also unitary. This ends the proof, since the converse statement is easily checked. □

References

[1] R. Abraham, J. Robbin, Transversal Mappings and Flows, W.A. Benjamin Inc., 1967.

[2] A.O. Barut, R. Raczka, Theory of Group representations and applications, World Scientific, second edition, 1986.

[3] B.E Boser, I.M. Guyon, V. Vapnik, "A Training Algorithm for Optimal Margin classifiers", proc. Fifth Ann. Workshop Computational Learning theory, ACM Press, 1992, pp. 144-152.

[4] S. Chen, A new vision system and the Fourier Descriptors Method by Group Representations, IEEE CDC conference, 1985, Las Vegas, USA.

[5] H. Chu, Compactification and duality of topological groups, Trans. American Math. Society, 123, 1966, pp. 310-324.

[6] J. Dixmier, Les C^*-algèbres et leurs repréentations, Gauthier-Villars, Paris, 1969.

[7] H. Fonga, Analyse harmonique sur les groupes et reconnaissance de formes, PHD Tthesis, université de Grenoble, 1992.

[8] K. Fukunaga, Introduction to statistical Pattern Recognition, second ed. Academic Press 1990.

[9] J.P. Gauthier, G. Bornard, M. Zilbermann, Harmonic analysis on motion groups and their homogeneous spaces, IEEE Trans on Systems, Man and Cyb., Vol 21, 1991.

[10] F. Ghorbel, A complete invariant description for grey level images by the harmonic analysis approach, Pattern Recognition Letters, 15, 1994.

[11] F. Gourd, Quelques Méthodes d'Analyse Harmonique en Théorie de la Détection et de la Reconnaissance de Formes, PHD thesis, Université de Grenoble, 1990.

[12] F. Gourd, J.P. Gauthier, H. Younes, Une méthode d'invariants de l'analyse harmonique en reconnaissance de formes, Traitement du Signal, Vol 6, N°3, 1989.

[13] E. Hewitt, K.A. Ross, Abstract Harmonic analysis, Springer Verlag Berlin, 1963.

[14] H. Heyer, Dualität über LokalKompakter Gruppen, Springer Verlag, 1970.

[15] A. Joyal, R. Street, An Introduction to Tannaka Duality and Quantum Groups, Lecture notes in maths 1488, Springer, pp. 411-492, 1991.
[16] A.B. Kyatkin, G.S. Chirikjan, Pattern matching as a correlation on the discrete Motion Group, Computer Vision and Image Understanding, Vol. 74, N°1, pp.22-35, 1999.
[17] Persoon, King Sun Fu, Shape discrimination using Fourier descriptors, IEEE systems man and cybernetics, SMC 7, N°3, 1977.
[18] F. Smach, C. Lemaître, J.P. Gauthier, J. Miteran, Generalized Fourier Descriptors with Applications to object recognition in SVM Context, 2006, Submitted.
[19] N. Tatsuuma, A duality Theorem for locally compact groups, J. Math. Kyoto Univ 6, 1967, pp. 187-293.
[20] J. Turski, Geometric Fourier Analysis of the conformal camera for active vision, SIAM review, Vol. 46, N°1, 2004.
[21] V. N. Vapnik, The statistical Learning Theory, Springer, 1998.
[22] N. Vilenkin, Fonctions spéciales et théorie de la représentation des groupes, Dunod, Paris, 1969.
[23] G. Warner, Harmonic Analysis on semi-simple Lie Groups, Springer Verlag, 1972.
[24] A. Weil, L'intégration dans les groupes topologiques et ses applications, Hermann, Paris, 1965.

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